# General Conditions for Global Intransitivities in Formal Voting Models Author(s): Richard D. McKelvey 

Source: Econometrica, Vol. 47, No. 5 (Sep., 1979), pp. 1085-1112
Published by: The Econometric Society
Stable URL: https://www.jstor.org/stable/1911951
Accessed: 12-08-2018 18:00 UTC

JSTOR is a not-for-profit service that helps scholars, researchers, and students discover, use, and build upon a wide range of content in a trusted digital archive. We use information technology and tools to increase productivity and facilitate new forms of scholarship. For more information about JSTOR, please contact support@jstor.org.

Your use of the JSTOR archive indicates your acceptance of the Terms \& Conditions of Use, available at https://about.jstor.org/terms

The Econometric Society is collaborating with JSTOR to digitize, preserve and extend access to Econometrica

# GENERAL CONDITIONS FOR GLOBAL INTRANSITIVITIES IN FORMAL VOTING MODELS ${ }^{1}$ 

By Richard D. McKelvey


#### Abstract

This paper proves that for majority voting over multidimensional alternative spaces, the majority rule intransitivities can generally be expected to extend to the whole alternative space in such a way that virtually all points are in the same cycle set. In other words, given almost any two points in the alternative space, it is possible to construct a majority path which starts at the first, and ends at the second. It is shown that for the intransitivities not to extend to the whole space in this manner, extremely restrictive conditions must be met on the frontier (or boundary) of the cycle set. Similar results are shown to hold for any social choice rule derived from a strong simple game. These results hold under fairly weak assumptions on individual preferences: individuals need only have continuous utility representations of their preferences such that no two individuals' preferences coincide locally. The results seem to rule out the possibility, at least in models of interest to economists, of using the transitive closure of the majority relation as a useful social choice function. They also imply that under any social choice rule meeting the conditions assumed here, it is generally possible to design agendas based on binary procedures which will arrive at virtually any point in the alternative space, even Pareto dominated points.


## 1. INTRODUCTION

Since Arrow's [1] pioneering work in the area, it has been known that for most social choice mechanisms, situations can arise in which the social ordering is intransitive even though all individuals hold transitive preferences. However, Arrow's theorem only tells us that there is some profile of individual preferences which can yield an intransitive social ordering. It does not tell us the likelihood with which we can expect such a situation to arise. Nor does it tell us the seriousness, or the extent of the intransitivities when they do occur. This paper deals with the above questions in the context of a particular class of social choice rules, namely those based on strong simple games, where the alternatives are a subset of a multidimensional space. Particular attention is given to the special case of majority rule. For such situations it is shown that not only will intransitivities usually arise, but also, the intransitivities will generally be global, so that all points in the space are members of the same cycle set.

The question of the likelihood with which intransitivities arise has already received a considerable amount of attention, especially for the case of majority rule. In fact, in multidimensional models of voting, our concern here, the conditions necessary just to guarantee transitivity at the top of the social ordering have been shown to be so severe that one would seldom expect them to be met in practice. Plott [17] has shown if all voters have continuous, differentiable utility representations of their preferences, that a necessary condition for the existence of a core point (i.e., a point that is undefeated under the majority relation) is that a very strong symmetry condition on individual gradient vectors be met. The condition is so strong that even if it were met, a minor perturbation of any one

[^0]voter's preferences would cause it to be violated. See Sloss [23], Davis, De Groot, and Hinich [5], and McKelvey and Wendell [14] for other versions of this result, and Matthews [12] and Slutsky [24] for extensions beyond simple majority rule. The generic nonexistence of a core also has been proven by Rubinstein [19] for the case when only continuity of preferences is assumed. Thus, existence of a core would seem to be a rare event. Further, transitivity at the top of the social ordering (i.e., existence of a core) does not guarantee anything about the rest of the social ordering. One is forced to conclude that the likelihood of obtaining a completely transitive social ordering in the case of majority rule would be extremely remote. Work of Kramer [10] and Schofield [20] on conditions for local transitivity reinforces this conclusion.

Although the difficulty of guaranteeing transitivity in multidimensional voting models is well known, it is not well understood how these intransitivities behave when transitivity breaks down. Thus, the question of the extent, or severity of the intransitivities is relatively unexplored. A substantial body of literature has developed recently under the implicit assumption that the intransitivities are fairly well behaved. This literature defines a derived social choice rule, called the transitive closure, which ranks two alternatives as socially indifferent if there is a cycle of which they both are members, and ranks $x$ better than $y$ if there is a finite path from $y$ to $x$ but not back. This effectively partitions the alternative space into "cycle sets," which are ordered transitively. The "top cycle set" is then of particular interest from both a normative and a descriptive point of view, the idea being that once alternatives in this set are proposed, society should not (or will not) then move to an alternative outside of the set. A review of this literature appears in Sen [22].

The usefulness of the above approach is of course dependent on the intransitivities in the social order being fairly limited in scope. Some recent research suggests that this hope may be unfounded. In a previous paper, I [13] have shown, in a model which assumes "Euclidian" preferences (i.e., preferences based on Euclidian distance from an individual ideal point), that when transitivity breaks down at all, it breaks down completely, so that all points in the policy space, $X$, are in the same cycle set. By this, it is meant that for any $x_{0}, y_{0} \in X$, it is possible to find a sequence $\theta_{0}, \ldots, \theta_{K} \in X$, with $\theta_{0}=x_{0}, \theta_{K}=y_{0}$, such that $\theta_{i+1}$ is preferred to $\theta_{i}$ by a majority for $1 \leqslant i \leqslant K-1$. Thus, it is possible to find a majority rule path between any two points in the space. Recently, Cohen [3], using methods of proof quite similar to those used in this paper, has shown that the result extends to the case when preferences are "elliptical," and has also shown uniqueness of the top cycle set for general convex preferences. The assumption of Euclidian, or even elliptical preferences is clearly quite restrictive. However, Schofield [20], using a very different approach, has shown a similar result in a model requiring only that preferences be continuous and differentiable. He shows that there usually exists a continuous majority rule path between any two points in the policy space. But Schofield has only shown this result for the case when the number of policy dimensions is large in relation to the number of voters. (He requires $m \geqslant q+1$, where $m$ is the dimensionality of the policy space, and $q$ is the number of voters in a minimal winning coalition.)

In this paper, it is shown that if the paths are not restricted to be continuous the above result extends to a very general model which places no restrictions on $m$. We assume only that voters have continuous utility representations of their preferences such that no two voters' preferences coincide locally. It is then shown that except under very restrictive conditions on individual preferences, global intransitivities will prevail. The conditions are such that if the alternative space, $X$, is any connected subset of $R^{2}$, one would usually expect them to fail, unless preferences are more or less linear over some region of $X$. If $X$ is any connected subset of $R^{m}$, with $m>2$, one would virtually always expect the conditions to fail, regardless of the nature of individual preferences. In fact, for $X \subseteq R^{m}$, with $m>2$, the conditions generally fail so badly that not only is there a majority path between any two points, but that path can be chosen in such a way that it is arbitrarily close to any pre-selected curve connecting the two points.

The above describes the situation for majority rule. We also look at the general class of social choice functions generated by strong simple games. The results are more difficult to interpret here. Although they seem somewhat less pessimistic than those for majority rule, they appear to be similar in spirit to those described above: Namely, unless there is one strong player, or a fortunate distribution of preferences, we would expect global intransitivities here too.

These results imply that for social choice rules meeting the conditions required here, the transitive closure would not in general be useful as a social choice function, since it would rank all alternatives as socially indifferent. The results also imply that in most cases social choice rules of the sort studied here would be subject to manipulation by anyone in control of the agenda. A clever agenda setter, with knowledge of all voter's preferences could design an agenda to reach virtually any point in the alternative space.

The rest of the paper is organized into four sections. The following section (Section 2) begins by giving notation and definitions. We define the set $P^{*}(x)$ as the set of points that are "reachable" from a point $x$ via the social relation. Section 3 then presents the main theorem, which proves that in order for $P^{*}(x)$ not to be the whole space, extremely restrictive conditions must be met on the "frontier" (or boundary) of $P^{*}(x)$. Section 4 interprets the results of Section 3 when utility functions are differentiable, and majority rule is in effect. It is proven that the main theorem then implies that for $P^{*}(x)$ not to be the whole space, an extremely strong symmetry condition on the gradients of individual utility functions must be met at all points on the boundary of $P^{*}(x)$. From this result one can see that the conditions would virtually always fail in any space of dimensionality greater than two. It also follows from this that the path can generally be chosen to follow any route desired. The final section discusses implications of these results, and gives some concluding remarks. The Appendix contains statements and proofs of a series of Lemmas used in the paper.

## 2. NOTATION AND DEFINITIONS

We assume a set of voters, $N=\{1,2, \ldots, n\}$, an alternative space $X$, which can be any topological space, although for illustrations we will assume $X \subseteq R^{m}$, and
we let $\Theta$ denote the set of binary relations over $X$. For each $i \in N$, we let $\Theta_{i} \subseteq \Theta$, be the set of possible preference relations for voter $i$ and we set $\bar{\Theta}=\Pi_{i \in N} \Theta_{i}$ to be the set of preference profiles over $X$. Elements of $\bar{\Theta}$ are written $\bar{R}=\left(R_{1}, \ldots, R_{n}\right)$, $\bar{R}^{\prime}=\left(R_{1}^{\prime}, \ldots, R_{n}^{\prime}\right)$, etc. For any binary relation $R \in \Theta$, we define two derived relations $P, I \in \Theta$ by $x P y \Longleftrightarrow(x R y$ and $\sim y R x)$, and $x I y \Longleftrightarrow(x R y$ and $y R x)$. Thus, for any $R_{i} \in \Theta_{i}$, the associated relations are written $P_{i}, I_{i}$. For convenience, we will also define a relation $Q$ by $x Q y \Longleftrightarrow y P x$, with similar definitions of $Q_{i}$ for individual voters.

A social welfare function is any function $f: \bar{\Theta} \rightarrow \bar{\Theta}$, which associates with each preference profile $\bar{R} \in \bar{\Theta}$, a relation $R=f(\bar{R})=f\left(R_{1}, \ldots, R_{n}\right) \in \Theta$. In this paper, we will consider only a special class of social welfare functions, namely those which are generated from strong simple games. To formalize this, any $C \subseteq N$ is called a coalition, with $|C|$ denoting the number of members of $C$. For any $C \subseteq N$, and $x, y \in X$, we write $x P_{c} y \Longleftrightarrow x P_{i} y$ for all $i \in C$. We let $\underline{W} \subseteq 2^{N}$ be any set of coalitions satisfying the following properties:
(2.1a) (Monotonicity) If $C \subseteq C^{\prime}$, and $C \in \underline{W}$, then $C^{\prime} \in \underline{W}$.
(2.1b) (Strong and Proper) $C \in \underline{W} \Longleftrightarrow N-C \notin W$.

Given any set of coalitions, $\underline{W}$, satisfying (a) and (b) above, we can define a social welfare function $R_{\underline{W}}=f_{\underline{W}}(\bar{R})$ as follows: For any $x, y \in X$,

$$
\begin{align*}
& x P_{\underline{W}} y \Longleftrightarrow x P_{C} y \text { for some } C \in \underline{W},  \tag{2.2}\\
& x R_{W} y \Longleftrightarrow \sim\left(y P_{W} x\right) .
\end{align*}
$$

The class of social welfare functions so generated (i.e., generated by a set of coalitions $\underline{W}$ satisfying (a) and (b)) will be denoted $\mathscr{F}$. Henceforth, we will only be concerned with social welfare functions in $\mathscr{F}$. (In general, we will drop the subscripts on $R_{\underline{W}}$ and $P_{\underline{w}}$, writing $R$ and $P$ for the social relations.) Note that if $n$ is odd, majority rule is in $\mathscr{F}$, where majority rule is defined by setting $\underline{W}=\underline{M}=$ $\{C \subseteq N||C|>n / 2\}$. If $n$ is even, then majority rule does not satisfy property 2 . However, modifications of majority rule, which break ties using a chairman's preference, would be in $\mathscr{F}$. In general, any weighted voting scheme, or representative system (which also breaks ties when all voters have strict preferences), will be in $\mathscr{F}$.

Now, for any $y, z \in X$, we use the notation

$$
\begin{equation*}
C_{y, z}=\left\{j \in N \mid y P_{j} z\right\} \tag{2.3}
\end{equation*}
$$

to represent the set of voters who prefer $y$ to $z$. For any $C \subseteq N$, we say voter $i$ is pivotal for $C$ if $C \notin \underline{W}$ and $C \cup\{i\} \in \underline{W}$. Then we have the following:

Definition 1: Let $y, z \in X$, and $i \in N$. Then $i$ is said to be a dummy voter with respect to $y$ and $x$ if, for any $C \subseteq N-\{i\}$ with

$$
C_{y, z}-\{i\} \subseteq C \subseteq N-C_{z, y},
$$

it is not the case that $i$ is pivotal for $C$, i.e.,

$$
C \cup\{i\} \in \underline{W} \Rightarrow C \in \underline{W} .
$$

If $i$ is not a dummy voter, he is said to be critical between $y$ and $z$. In this case, there is a $C \subseteq N-\{i\}$ with $C_{y, z}-\{i\} \subseteq C \subseteq N-C_{z, y}$ such that $C \cup\{i\} \in \underline{W}$ and $C \notin \underline{W}$.

Thus, a voter is a dummy voter if, no matter how the indifferent votes are cast, the voter has no chance of affecting the outcome. A critical voter, on the other hand, is a voter whose vote is worth something, in the sense that there is some reassignment of preferences to the indifferent voters such that the voter in question becomes pivotal.

Definition 2: Let $y, z \in X$, and $i, j \in N$; then voter $i$ is said to be as strong as voter $j$ between $y$ and $z$ if, for every $C \subseteq N-\{i, j\}$ with

$$
C_{y, z}-\{i, j\} \subseteq C \subseteq N-C_{z, y},
$$

$j$ pivotal for $C \Rightarrow i$ pivotal for $C$, i.e.,

$$
C \cup\{j\} \in \underline{W} \Rightarrow C \cup\{i\} \in \underline{W}
$$

Note that if $y I_{i} z$ and $y I_{i} z$, then if voter $j$ is not a dummy voter between $y$ and $z$, and voter $i$ is as strong as voter $j$, then $i$ is not a dummy voter between $y$ and $z$.

Now for any $S \subseteq X$, we let $S^{0}$ denote the interior of $S, \bar{S}$ denote the closure of $S$, $S^{c}$ denote the complement of $S$, and $\underline{B}(S)$ denote the boundary of $S$ (i.e., $\left.\underline{B}(S)=\bar{S} \cap\left(\overline{\boldsymbol{S}^{c}}\right)\right)$. Then, we define the frontier of $S, \underline{F}(S)$, as

$$
\begin{equation*}
\underline{F}(\boldsymbol{S})=\left(\overline{\boldsymbol{S}^{\boldsymbol{o}}}\right) \cap\left(\overline{\left.\boldsymbol{S}^{c}\right)^{\boldsymbol{o}}}\right) \tag{2.4}
\end{equation*}
$$

All of these definitions are in the topology on $X$. Note that for any $S \subseteq X^{-} \underline{F}(S)$ and $\underline{B}(S)$ are always closed sets, with $\underline{F}(S) \subseteq \underline{B}(S)$. The frontier of a set $S$, then, is simply a subset of the boundary of $S$, consisting of all points that are arbitrarily close to the interior of $S$ and the interior of its complement. See Figure 2.1 for an illustration when $S \subseteq X \subseteq R^{2}$. In this figure $S$ includes the line protruding from the


Figure 2.1-Illustration of $\underline{B}(S)$ and $\underline{F}(S)$ ( $S$ includes protruding line and intruding line).
main body of $S$, but does not include the line going in. The frontier of $S$ consists only of the heavy line around $S$, while the boundary also includes the protruding and intruding lines.

Next, we define, for any binary relation $R \in \Theta$ and $x \in X$, a correspondence $R^{i}: X \rightarrow 2^{X}$ by

$$
\begin{align*}
& R^{1}(x)=R(x)=\{y \in X \mid y R x\}  \tag{2.5}\\
& R^{i}(x)=\left\{y \in X \mid y R z \text { for some } z \in R^{i-1}(x)\right\},
\end{align*}
$$

and

$$
R^{*}(x)=\bigcup_{i=1}^{\infty} R^{\prime}(x)
$$

Thus, $R^{i}(x)$ is the set of points in $X$ which can be reached in $j$ steps, via the relation $R$, starting at $x . R^{*}(x)$ is the set of points which can be reached in some finite number of steps via the relation $R$ (or in 1 step via the relation $R^{*}$, where $R^{*}$ is the transitive closure of $R$ ). If $R=f(\bar{R})$, where $f \in \mathscr{F}$, and $P$ and $I$ are associated strong and equivalence relations, then $R^{i}(x), P^{i}(x)$, and $I^{i}(x)$ are the sets of points which can be reached in $j$ steps via $R, P$, and $I$, respectively. Similarly, $R_{i}^{i}(x)$, $P_{i}^{i}(x)$, and $I_{i}^{i}(x)$ are the sets of points that can be reached in $j$ steps via the individual relations $R_{i}, P_{i}$, and $I_{i}$. Note that if the relation $R_{i}$ is transitive, that $R_{i}^{i}(x)=R_{i}^{k}(x)$ for all $j, k$, so in this case $R_{i}(x)=R_{i}^{*}(x)$.

## 3. THE MAIN RESULTS

The main object of interest in this paper is the set $P^{*}(x)$. This is the set of points which are reachable, by some finite path, via the social relation, $P$. In other words, for any point $y \in P^{*}(x)$, there is an integer $K>0$ and a sequence $\left\{\theta_{i}\right\}_{i=0}^{K}$, with $\theta_{0}=x, \theta_{K}=y$, and $\theta_{i} P \theta_{i-1}$ for all $1 \leqslant i \leqslant K$. We want to determine how big $P^{*}(x)$ is, for arbitrary $x \in X$. In order to investigate this question, we do not investigate it directly, but rather look at properties that are satisfied on the frontier of $P^{*}(x)$. It will be shown that in general, very restrictive conditions must be met on $\underline{F}\left(P^{*}(x)\right)$. For many social choice functions-in particular for majority rule-the conditions that must be satisfied on $\underline{F}\left(P^{*}(x)\right)$ are so restrictive as to imply that the frontier will be empty. But if $X$ is connected, $\underline{\underline{F}}\left(P^{*}(x)\right)=\varnothing$ implies (via Lemma 5 of the. Appendix), that either $P^{*}(x)=\varnothing$ or $\overline{P^{*}(x)}=X$. The first possibility corresponds to the case where $x$ is a core point (i.e., $y P x$ for no $y \in X$ ), and it is known from Plott's theorem [17] that the conditions for a core point are unlikely to be met in practice. This leaves the remaining possibility, namely $\overline{P^{*}(x)}=X$ as the situation that would be expected in general. Of course $P^{*}(x)=X$ implies that virtually any point in the entire space is reachable from $x$.

Thus, the question of the size of $P^{*}(x)$ reduces to the question of the existence of its frontier. If $\underline{F}\left(P^{*}(x)\right)=\varnothing$, we can conclude that almost any point in $X$ can be
reached from $x$. The rest of this paper, then, will be devoted to studying the frontier of $P^{*}(x)$. This section studies the "global" properties of $\underline{F}\left(P^{*}(x)\right)$, while Section 4 looks at the local properties of $\underline{F}\left(P^{*}(x)\right)$.

Throughout the remainder of the paper, we assume that each individual has a continuous utility representation of his preferences, and that he has no "flat spots," or regions of indifference in his preferences. Formally, we make the following two assumptions.

Assumption 1: For each $i \in N$, there is a continuous function $u_{i}: X \rightarrow R$, satisfying, for all $x, y \in X, u_{i}(x) \geqslant u_{i}(y) \Longleftrightarrow x R_{i} y$.

Assumption 2: For each $i \in N$, and all $y \in X,\left(I_{i}(y)\right)^{0}=\left\{x \in X \mid x I_{i} y\right\}^{0}=\varnothing$.
It should be noted that although Assumption 2 explicitly makes restrictions only on individual preferences, it also implicitly puts some restrictions on $X$.

For example, $X$ could not be a finite alternative set, for then it follows from Assumption 1 that regardless of the topology on $X$, that $I_{i}(y)$ is open, ${ }^{2}$ hence $\left(I_{i}(y)\right)^{0} \neq \varnothing$. Similarly, if $X \subseteq R^{m}$, and $R^{m}$ has the usual topology, then $X$ can contain no isolated points, for if $y \in X$ is an isolated point, then in the relative topology on $X, y \in\left(I_{i}(y)\right)^{0}$.

With the above two assumptions, a number of results about the properties of preference sets for individuals and for the social relation can be proven. These are formally stated and proven in Lemmas 2-4 of the Appendix. Using these results, we prove the following theorem which gives conditions that must be satisfied by the set $P^{*}(x)$.

Theorem 1: If each voter satisfies Assumption 1, then for any $x \in X$, and all $y \in \underline{B}\left(P^{*}(x)\right)$,

$$
P(y) \subseteq \overline{P^{*}(x)} \subseteq R(y)
$$

If all voters also satisfy Assumption 2, then

$$
\overline{P(y)}=\overline{P^{*}(x)} .
$$

It follows that $\underline{F}(P(y))=\underline{F}\left(P^{*}(x)\right)$.

Proof: First we deal with the case when only Assumption 1 is met and prove $P(y) \subseteq \overline{P^{*}(x)}$. Assume the contrary. Then, we set $G=P(y)-\overline{P^{*}(x)}$. Since $P(y)$ is open (by Lemma 4a), it follows that $G$ is open and non-empty, and $P(y) \cap G \neq \varnothing$. Thus, by lower semi-continuity of $P(x)$ (Lemma 4b), there is a neighborhood $N(y)$ of $y$ such that for all $z \in N(y), P(z) \cap G \neq \varnothing$. But since $y \in B\left(P^{*}(x)\right)$, it follows that $N(y) \cap P^{*}(x) \neq \varnothing$. Hence, pick $z^{*} \in N(y) \cap P^{*}(x)$. Then since

[^1]$P\left(z^{*}\right) \cap G \neq \varnothing$, pick $w^{*} \in P\left(z^{*}\right) \cap G$. Now $z^{*} \in P^{*}(x)$ and since $w^{*} \in P\left(z^{*}\right)$, it follows that $w^{*} \in P^{*}(x)$. This is a contradiction since $w^{*} \in G$, and $G \cap P^{*}(x)=\varnothing$. So, we must have $P(y) \subseteq P^{*}(x)$.

Now, to prove $\overline{P^{*}(x)} \subseteq R(y)$ it suffices to prove $P^{*}(x) \subseteq R(y)$, since $R(y)$ is closed (that $R(y)$ is closed follows from Lemma 3a, since $R(y)=(Q(y))^{c}$, and $Q(y)$ is open). We assume it is not the case that $P^{*}(x) \subseteq R(y)$. Then let $z \in P^{*}(x)$ with $z \notin R(y)$. It follows that $y P z \Rightarrow y \in P^{*}(x)$. But this is a contradiction since by assumption, $y \in \underline{B}\left(P^{*}(x)\right)$, and since $P^{*}(x)$ is open, $y \notin P^{*}(x)$. Thus, $P^{*}(x) \subseteq R(y)$, and we are done with the first part of the theorem.

Now, we assume all $u_{i}$ satisfy Assumption 2 and prove $\overline{P(y)}=\overline{P^{*}(x)}$. Since $P(y) \subseteq \overline{P^{*}(x)}$ from the above proof, it follows that $\overline{P(y)} \subseteq \overline{P^{*}(x)}$. Hence we need only show that $\overline{P^{*}(x)} \subseteq \overline{P(y)}$. In fact it is sufficient to show $P^{*}(x) \subseteq \overline{P(y)}$. To show this, assume $z \in P^{*}(x)$ and $z \notin \overline{P(y)}$. Then $z \notin \overline{P(y)} \Rightarrow z \notin P(y) \Rightarrow z \in Q(y)$ or $z \in$ $I(y)$. But $z \in Q(y)$ is impossible, since $y P z$ and $z \in P^{*}(x)$ implies $y \in P^{*}(x)$, a contradiction, so we must have $z \in I(y)$. But then by Lemma 3b, it follows that either $z \in \overline{P(y)}$ or $z \in \overline{Q(y)}$. By assumption, the former does not hold, so $z \in \overline{Q(y)}$. Hence, in any neighborhood of $z$, we can find a point $z^{*} \in Q(y)$. Since $P^{*}(x)$ is open (by Lemma 4a), and $z \in P^{*}(x)$, we can pick $z^{*} \in Q(y) \cap P^{*}(x)$. In other words, $y P z^{*}$, with $z^{*} \in P^{*}(x)$. It follows that $y \in P^{*}(x)$. However, again this is a contradiction, since by assumption that $y \in \underline{B}\left(P^{*}(x)\right)$, it follows that $y \notin P^{*}(x)$. Hence $z \in \overline{P(y)}$, and $\overline{P(y)}=\overline{P^{*}(x)}$.
Q.E.D.

Thus, under Assumptions 1 and 2, Theorem 1 shows that it must be the case that for any point, $y$, on the boundary of $P^{*}(x)$, the set of points which can be reached in one step from $y$ must coincide with $P^{*}(x)$, with the possible exception of points of closure.

This result implies restrictions on individual utility functions at frontier points of $P^{*}(x)$, which will be the subject of the next theorem. To obtain these implications, we first need an additional assumption.

Assumption 3 (Diversity of Preferences): For all open $S \subseteq X, y \in \underline{F}(S)$, and $i$, $j \in N, I_{i}(y) \cap I_{i}(y)$ has no interior in the relative topology on $\underline{F}(S)$.

This assumption guarantees that no two voters have preferences whose indifference contours exactly coincide locally. To understand the assumption, consider the case when $X \subseteq R^{m}$. Then for an open set $S \subseteq X, \underline{F}(S)$ can be thought of as defining an arbitrary $n-1$ dimensional manifold in $X$. The assumption then states that no two voters can have indifference contours which coincide on any open subset of such a manifold. Note that this does not preclude two indifference contours from crossing or being tangent at a point.

Assumption 3 would be met if all voters had "Euclidian" preferences (i.e., preferences based on Euclidian distance from some "ideal point"), as long as the ideal points of all voters were distinct. Also, as Cohen [3] proves, the assumption is met if all voters have "elliptical" preferences as long as no two voters' preferences are exactly the same over the entire space $\boldsymbol{X}$. Finally note that Assumption 3 implies Assumption 2, so that Assumption 2 is redundant, given Assumption 3.

In the following theorem, we are concerned with a particular subset of an individual's indifference set, which we call the "indifference frontier." For any $i \in N$ and $y \in X$, we define the indifference frontier $\underline{I F}_{i}(y)$ for voter $i$ through $y$ by

$$
\begin{equation*}
\underline{I F}_{i}(y)=\underline{F}\left(P_{i}(y)\right) \cap \underline{F}\left(Q_{i}(y)\right) . \tag{3.1}
\end{equation*}
$$

Under Assumptions 1 and 2, it follows (by Lemma 2c of the Appendix) that

$$
\begin{equation*}
\underline{I F}_{i}(y)=\underline{F}\left(P_{i}(y)\right)=\underline{F}\left(Q_{i}(y)\right) \subseteq I_{i}(y) . \tag{3.2}
\end{equation*}
$$

Thus, the indifference frontier of voter $i$ through $y$ is a subset of the indifference contour of voter $i$ through $y$, which coincides, under our assumptions, with the frontier of the set of points he prefers to $y$.

Further, for any $x \in X$, we let $Y_{x}=\underline{F}\left(P^{*}(x)\right)$, and we use the notation $\hat{P}_{i}=P_{i} / Y_{x}$ to denote the relation $P_{i}$ restricted to $Y_{x}$. $\hat{Q}_{i}$ is defined similarly. We then define the indifference frontier relative to $Y_{x}$, for voter $i \in N$, and $y \in Y_{x}$ by

$$
\begin{equation*}
\underline{I \hat{F}}_{i}(y)=\underline{F}\left(\hat{P}_{i}(y)\right) \cap \underline{F}\left(\hat{Q}_{i}(y)\right) \tag{3.3}
\end{equation*}
$$

where the frontiers are defined in the relative topology on $Y_{x}$. It will follow from Assumptions 1-3 together with part 1 of the following theorem that for all but one voter, say voter $j$, Assumptions 1 and 2 are satisfied on $\underline{F}\left(P^{*}(x)\right)$, while voter $j$ is indifferent between all points in $\underline{F}\left(P^{*}(x)\right)$. Hence, as above, we can write, for all $i \in N$,

$$
\begin{equation*}
\underline{I}_{i}(y)=\underline{F}\left(\hat{P}_{i}(y)\right)=\underline{F}\left(\hat{Q}_{i}(y)\right) . \tag{3.4}
\end{equation*}
$$

The set $\underline{I}_{i}(y)$ can be thought of as voter $i$ 's indifference frontier through $y$ relative to his preferences on $\underline{F}\left(P^{*}(x)\right)$. Equivalently, it can be thought of as the set of points where individual $i$ 's indifference frontier crosses $\underline{F}\left(P^{*}(x)\right)$.

With these definitions and assumptions, we can now prove the main theorem of this paper.

Theorem 2: Assume all voters satisfy Assumptions 1 and 2 and 3, and let $x \in X$; then:
(i) There is some $j \in N$, such that for all $y \in \underline{F}\left(P^{*}(x)\right), \underline{F}\left(P^{*}(x)\right) \subseteq \underline{I F_{j}}(y) \subseteq I_{j}(y)$.
(ii) Let $y, z \in \underline{F}\left(P^{*}(x)\right.$ ), and $z \in \underline{I F}_{i}(y)$ for some $i \in N-\{j\}$. Then (a) if $i$ is not a dummy voter with respect to $y$ and $z, \exists k \in N-\{i, j\}$ with $z \in I_{k}(y)$; (b) if i is as strong as $j$ with respect to $y$ and $z, \exists k \in N-\{i, j\}$ with $z \in \underline{I}_{k}(y)$.

Proof: We prove (i) first. The result is trivially true if $\underline{F}\left(P^{*}(x)\right)=\varnothing$, so assume $\underline{F}\left(P^{*}(x)\right) \neq \varnothing$. By Theorem 1, Lemma 3c, and Lemma 3d, it follows that for any $y \in \underline{F}\left(P^{*}(x)\right)$,

$$
\begin{equation*}
\underline{F}\left(P^{*}(x)\right)=\underline{F}(P(y)) \subseteq I(y) \subseteq \bigcup_{i \in N} I_{i}(y) . \tag{3.5}
\end{equation*}
$$

We define, for any $y \in \underline{F}\left(P^{*}(x)\right)$,

$$
\begin{equation*}
V_{i}(y)=\underline{F}\left(P^{*}(x)\right) \cap I_{i}(y) . \tag{3.6}
\end{equation*}
$$

Thus,

$$
\begin{equation*}
\bigcup_{i \in N} V_{i}(y)=\underline{F}\left(P^{*}(x)\right) . \tag{3.7}
\end{equation*}
$$

Further, since each $I_{i}(y)$ is closed, it follows that each $V_{i}(y)$ is closed in the relative topology on $\underline{F}\left(P^{*}(x)\right)$.

We first want to show that for any $y_{0} \in \underline{F}\left(P^{*}(x)\right), V_{j}\left(y_{0}\right)=\underline{F}\left(P^{*}(x)\right)$ for some $j \in N$. Clearly $\underline{F}\left(P^{*}(x)\right)$ has a relative interior, since $\underline{F}\left(P^{*}(x)\right)$ is non-empty and open in the relative topology on $\underline{F}\left(P^{*}(x)\right)$. So, from (3.7) and Lemma 1, it follows that some $V_{i}\left(y_{0}\right)$ must have an interior in the relative topology on $\underset{F}{ }\left(P^{*}(x)\right)$. We assume, without loss of generality, that $V_{j}\left(y_{0}\right)$ has a non-empty interior. We will then show that $V_{i}\left(y_{0}\right)=\underline{F}\left(P^{*}(x)\right)$. To show this, we construct a sequence of alternatives, $y_{1}, \ldots, y_{n} \in \underline{F}\left(P^{*}(x)\right)$, and a sequence of subsets $W_{1}, \ldots, W_{n} \subseteq$ $\underline{F}\left(P^{*}(x)\right)$ as follows:

$$
\begin{equation*}
W_{1}=V_{j}\left(y_{0}\right) \tag{3.8}
\end{equation*}
$$

Then, if $W_{k}$ has a non-empty interior, we construct $y_{k}$ and $W_{k+1}$ as follows:

$$
\begin{align*}
& y_{k} \in W_{k}  \tag{3.9}\\
& W_{k+1}=W_{k}-\bigcup_{i \neq i} V_{i}\left(y_{k}\right) .
\end{align*}
$$

If $W_{k}$ has a non-empty interior, it follows that $W_{k+1}$ has a non-empty interior. To see this, we note first that $y_{k} \in W_{k} \subseteq W_{1}=V_{i}\left(y_{0}\right)$. But by transitivity of $I_{j}, V_{j}\left(y_{k}\right)=V_{i}\left(y_{0}\right)$. Thus, $W_{k} \subseteq V_{j}\left(y_{k}\right)$ and we can rewrite $W_{k+1}$ as

$$
\begin{align*}
W_{k+1} & =W_{k}-\bigcup_{i \neq i}\left(V_{i}\left(y_{k}\right) \cap V_{i}\left(y_{k}\right)\right)  \tag{3.10}\\
& =W_{k}-A
\end{align*}
$$

where $A=\bigcup_{i \neq j} A_{i}$, and $A_{i}=V_{i}\left(y_{k}\right) \cap V_{i}\left(y_{k}\right)$. Clearly, each $A_{i}$ is closed in the relative topology on $\underline{F}\left(P^{*}(x)\right.$ ), and by Assumption 3, it follows that each $A_{i}$ has no interior. Thus, $\boldsymbol{A}$ is closed, and by Lemma 1 , has no interior. But now, since $W_{k}$ has a non-empty interior, there is a non-empty open set $B \subseteq W_{k}$. Since we cannot have $B \subseteq A$, it follows that $C=B-A$ is non-empty and open and $C \subseteq W_{k+1}$. Thus $W_{k+1}$ has a non-empty interior, as we wished to show. Hence, by induction, it follows that we can construct a sequence of alternatives $y_{1}, \ldots, y_{n} \in \underline{F}\left(P^{*}(x)\right)$ and of sets $W_{1}, \ldots, W_{n} \subseteq \underline{F}\left(P^{*}(x)\right)$ satisfying (3.9) for all $k \in N$.

It is easily verified that for any $r, s, k \in N$ with $k \neq j$, and $r \neq s$, the following two properties are satisfied:

$$
\begin{equation*}
V_{k}\left(y_{r}\right) \cap V_{k}\left(y_{s}\right)=\varnothing \tag{3.11}
\end{equation*}
$$

and

$$
\begin{equation*}
\bigcup_{i \in N} V_{i}\left(y_{r}\right)=\bigcup_{i \in N} V_{i}\left(y_{s}\right)=\underline{F}\left(P^{*}(x)\right) . \tag{3.12}
\end{equation*}
$$

The second property follows directly from (3.7). To see that the first property is satisfied, assume, without loss of generality, that $r>s$. Then $y_{r} \in W_{r} \subseteq W_{s+1}=$ $W_{s}-\bigcup_{i \neq i} V_{i}\left(y_{s}\right) \Rightarrow y_{r} \notin V_{k}\left(y_{s}\right)$. Thus, by transitivity of $I_{k}$, we get (3.11).

Now to show that $V_{i}\left(y_{0}\right)=\underline{F}\left(P^{*}(x)\right)$, we assume this is not the case. Then $\underline{F}\left(P^{*}(x)\right)-V_{j}\left(y_{0}\right) \neq \varnothing$, so we pick $y^{*} \in \underline{F}\left(P^{*}(x)\right)-V_{j}\left(y_{0}\right)$. By (3.12) it follows that for each $r \in N$ there is a $k_{r} \in N-\{j\}$ such that

$$
\begin{equation*}
y^{*} \in V_{k_{r}}\left(y_{r}\right) \tag{3.13}
\end{equation*}
$$

But, then for some $r, s \in N, k_{r}=k_{s}$; i.e.,

$$
\begin{equation*}
y^{*} \in V_{k_{r}}\left(y_{r}\right) \cap V_{k_{r}}\left(y_{s}\right) \tag{3.14}
\end{equation*}
$$

But this is a contradiction to (3.11), hence $\underline{F}\left(P^{*}(x)\right)-V_{i}\left(y_{0}\right)=\varnothing \Rightarrow \underline{F}\left(P^{*}(x)\right)=$ $V_{i}\left(y_{0}\right)$ as we wished to show.

It follows that $\underset{F}{ }\left(P^{*}(x)\right) \subseteq I_{i}(y)$, for some $j \in N$. Now, we must show that $\underline{F}\left(P^{*}(x)\right) \subseteq I F_{i}(y) \subseteq I_{j}(y)$. That $I F_{j}(y) \subseteq I_{i}(y)$ follows directly from Lemma 2c. Now if it is not the case that $\underline{F}\left(P^{*}(x)\right) \subseteq \underline{I F_{j}}(y)$, then $\underline{F}\left(P^{*}(x)\right)-\underline{I F_{j}}(y)$ is nonempty and open in the relative topology on $\underset{F}{ }\left(P^{*}(x)\right)$. But by Lemma 3 e $\underline{\underline{F}}\left(P^{*}(x)\right) \subseteq \bigcup_{i \in N} \underline{F}\left(P_{i}(y)\right)=\bigcup_{i \in N} \underline{I F}_{i}(y)$. It follows that for some $i \neq j, \underline{F_{i}}(y)$ has an interior in the relative topology on $\underline{F}\left(P^{*}(y)\right)$. But then $\left(I_{i}(y) \cap I_{i}(y)\right)$ also has an interior in the relative topology on $\underline{F}\left(P^{*}(x)\right)$, a contradiction to Assumption 3. Thus we must have $\underline{F}\left(P^{*}(x)\right) \subseteq \underline{F}\left(P_{i}(y)\right) \subseteq I_{i}(y)$, and (i) is proven.

To prove (ii)(a), let $y, z \in \underline{F}\left(P^{*}(x)\right)$, and assume $i \in N-\{j\}$ is not a dummy voter with respect to $y$ and $z$, and let $z \in \underline{I}_{i}(y)$. Assume the consequence of ii(a) is false. Then pick an open neighborhood $N(z)$ of $z$ such that, for all $k \notin\{i, j\}$, either

$$
\begin{equation*}
N(z) \subseteq P_{k}(y) \quad \text { or } \quad N(z) \subseteq Q_{k}(y) \tag{3.15}
\end{equation*}
$$

Since $z \in \hat{I F}_{i}(y)$, it follows that the following two sets are non-empty:

$$
\begin{align*}
& A_{1}=\underline{F}\left(P^{*}(x)\right) \cap P_{i}(y) \cap N(z)  \tag{3.16}\\
& A_{2}=\underline{F}\left(P^{*}(x)\right) \cap Q_{i}(y) \cap N(z) .
\end{align*}
$$

So pick $w_{1} \in A_{1}$, and $w_{2} \in A_{2}$. Since $w_{1}, w_{2} \in \underline{F}\left(P^{*}(x)\right)$, and $\underline{F}\left(P^{*}(x)\right)=\underline{F}(P(y)) \subseteq$ $I(y)$ (by Theorem 1 and Lemma 3c), it follows that $w_{1} \notin P(y)$ and $w_{2} \notin Q(y)$. Thus, it follows that
(3.17) $\quad C_{w_{1}, y} \notin \underline{W}$ and $C_{v, w_{2}} \notin \underline{W}$,
i.e., $C_{z, y} \cup\{i\} \notin \underline{W}$ and $C_{y, z} \cup\{i\} \notin \underline{W}$, which implies, via (2.1b),

$$
\begin{equation*}
C_{z, v} \cup\{j\} \in \underline{W} . \tag{3.18}
\end{equation*}
$$

But, since the game is strong, we have

$$
\begin{align*}
& C_{z, y}=C_{z, y}-\{i\} \notin \underline{W}, \quad \text { and } \quad C_{z, y} \cup\{i\} \notin \underline{W},  \tag{3.19}\\
& C_{z, y} \cup\{j\}=C_{z, y} \cup\{j\}-\{i\} \in \underline{W}, \quad \text { and } \\
& C_{z, y} \cup\{j\} \cup\{i\} \in \underline{W} .
\end{align*}
$$

And since voters $i$ and $j$ are the only voters who do not hold strong preferences
between $y$ and $z$, it follows that voter $i$ is a dummy voter with respect to $y$ and $z$. Hence, we have a contradiction, and it follows that $z \in I_{k}(y)$ for some $k \notin\{i, j\}$ so $\mathrm{ii}(\mathrm{a})$ is proven.

Now to prove ii(b), let $y, z \in \underline{F}\left(P^{*}(x)\right)$, with $z \in \underline{I}_{i}(y)$ for $i \in N-\{j\}$, and assume $i$ is as strong as $j$ with respect to $y$ and $z$. Then assume $z \notin \underline{\hat{F}_{k}}(y)$ for all $k \in N-\{i, j\}$. Then there is a neighborhood $N(z)$ of $z$ such that $N(z) \cap \underline{\underline{I}}_{k}(y)=\varnothing$ for all $k \in N-\{i, j\}$. It follows that each of the following sets is nonempty:

$$
\begin{align*}
& A_{1}=N(z) \cap Q_{i}(y) \cap \underline{F}\left(P^{*}(x)\right)-\bigcup_{k \in N-\{i, i\}} I_{k}(y),  \tag{3.20}\\
& A_{2}=N(z) \cap P_{i}(y) \cap \underline{F}\left(P^{*}(x)\right)-\bigcup_{k \in N-\{i, j\}} I_{k}(y) .
\end{align*}
$$

This follows because the sets $N(z) \cap Q_{i}(y) \cap \underline{F}\left(P^{*}(x)\right)$ and $N(z) \cap P_{i}(y) \cap$ $\underline{F}\left(P^{*}(x)\right)$ are both non-empty (since $z \in \underline{\underline{I}}(y)$ ) and open in the relative topology on $\underline{F}\left(P^{*}(x)\right)$. But then letting $A=A_{1} \cup A_{2}$, it follows by construction that for all $k \in N-\{i, j\}$ that $A \subseteq P_{k}(y)$ or $A \subseteq Q_{k}(y)$. Now pick $w_{1} \in A_{1}, w_{2} \in A_{2}$; since $w_{1} \in \underline{F}\left(P^{*}(x)\right) \subseteq \underline{F}(P(y)) \subseteq I(y)$ and $w_{2} \in \underline{F}\left(P^{*}(x)\right) \subseteq I(y)$, it follows that

$$
\begin{equation*}
C_{w_{1}, y} \notin W \quad \text { and } \quad C_{w_{2}, y} \notin W \tag{3.21}
\end{equation*}
$$

By construction $C_{w_{2}, y}=C_{w_{1}, y} \cup\{i\}$. Further, $C_{w_{1}, y} \cup\{j\}=N-C_{w_{2}, y}$, so, since $\underline{W}$ is generated by a strong game, $C_{w_{1}, y} \cup\{j\} \in \underline{W}$. Setting $C=C_{w_{1}, v}$, we have shown

$$
\begin{equation*}
C \cup\{i\} \notin \underline{W}, \quad \text { but } \quad C \cup\{j\} \in \underline{W} \tag{3.22}
\end{equation*}
$$

where $C_{z, y}-\{i, j\}=C_{z, y} \subseteq C \subseteq N-C_{y, z}$. In other words, it is not the case that $i$ is as strong as $j$ with respect to $y$ and $z$, which is a contradiction. Hence, $z \in \underline{I}_{k}(y)$ for some $k \in N-\{i, j\}$.
Q.E.D.

For the case when $R=f(\bar{R})$ is generated by majority rule, condition (ii) of the above theorem can be simplified, as in this case all voters are as strong as $j$. We thus get the following corollary to Theorem 2 for majority rule.

Corollary 1: Assume all voters satisfy Assumptions 1, 2, and 3, assume $R=f(\bar{R})$ is generated by majority rule, with $n$ odd, and iet $x \in X$; then:
(i) there is some $j \in N$ such that for all $y \in \underline{F}\left(P^{*}(x)\right)$

$$
\underline{F}\left(P^{*}(x)\right) \subseteq \underline{I F_{i}}(y) \subseteq I_{j}(y) ;
$$

(ii) for all $y \in \underline{F}\left(P^{*}(x)\right)$, and all $i \in N$,

$$
\underline{I}_{i}(y) \subseteq \bigcup_{k \in N-\{i, j\}} \underline{I}_{k}(y) .
$$

Proof: This follows directly from Theorem 2 with the observation that if $R=f(\bar{R})$ is generated by majority rule, then for any $y, z \in X$, and $i, j \in N$, voter $i$ is as strong as voter $j$. This is true because for any $C \subseteq N-\{i, j\},|C \cup\{j\}|=|C \cup\{i\}|$, hence $C \cup\{j\} \in \underline{W} \Leftrightarrow C \cup\{i\} \in \underline{W}$.
Q.E.D.

We now interpret the above theorem and corollary. The conditions (i) and (ii) of the theorem give conditions that must be met by all points on the frontier of $P^{*}(x)$. Condition (i) requires that the frontier of $P^{*}(x)$ must be a subset of an indifference contour for some voter. In other words, there is one voter, who we label voter $j$, who is indifferent between all points on $\underline{F}\left(P^{*}(x)\right)$. Condition (ii) (a) of the theorem says that for any other voter, say voter $i$, if his indifference curve through a point $y \in \underline{F}\left(P^{*}(x)\right)$ crosses $\underline{F}\left(P^{*}(x)\right)$ at $z$, then as long as he is not a dummy voter between $y$ and $z$, there must be another voter, say voter $k$, whose indifference set through $y$ also passes through $z$. Condition (ii) (b) is simply a modification of (ii) (a), which guarantees that if voter $i$ is as strong as $j$, then voter $k$ 's indifference contour must also cross $\underline{F}\left(P^{*}(x)\right)$ at $z$. Note that for majority rule, all voters are as strong as $j$, hence condition (ii) of the corollary requires that for any voter $i \in N$, if voter $i$ 's indifference frontier through $y$ crosses through a point $z \in \underline{F}\left(P^{*}(x)\right)$, then there must be at least one other voter whose indifference contour through $y$ also crosses through $z$.

In order to illustrate the above results, we distinguish three possible cases:

Case I: $\underline{F}\left(P^{*}(x)\right)=\varnothing$.
CASE II: $\underline{F}\left(P^{*}(x)\right) \neq \varnothing$, but $\quad \hat{I}_{i}(y) \subseteq\{y\}$ for all $i \in N, y \in \underline{F}\left(P^{*}(x)\right)$.
CASE III: $\underline{F}\left(P^{*}(x)\right) \neq \varnothing \quad$ and $\quad \underline{I}_{i}(y)-\{y\} \neq \varnothing$ for some $y \in \underline{F}\left(P^{*}(x)\right), i \in N$.

The first case is the case when the frontier of $P^{*}(x)$ is empty and, as we shall see, is the situation we would expect in general. In Case II, the frontier of $P^{*}(x)$ is not empty, but each individual indifference frontier crosses $\underline{F}\left(P^{*}(x)\right)$ at most once. Finally, in Case III, $\underline{F}\left(P^{*}(x)\right)$ is not empty, and at least one voter has an indifference frontier that crosses $\underline{F}\left(P^{*}(x)\right)$ in at least two points.

Now in Case I, when $\underline{F}\left(P^{*}(x)\right)=\varnothing$, both conditions (i) and (ii) of the corollary (also of the theorem) are met vacuously. In Case II, condition (ii) of the corollary (theorem) is met vacuously, although condition (i) is not. An illustration of this case when $X \subseteq R^{2}$ and $n=3$ is given in Figure 3.1. Here, the frontier of $P^{*}(x)$ must coincide with one voter's indifference frontier, and all indifference frontiers for all other voters can cross this frontier only once. Note that Case II can only occur with particular types of preferences, when dimension of the space is small. For example, if $X=R^{m}$, with $m \geqslant 2$, and $P_{i}(x)$ is bounded for all $i \in N, x \in X$, then Case II could not occur. Further, Case II can only occur if $m \leqslant 2$. Specifically, for $m \geqslant 3$, then $\underline{F}\left(P^{*}(x)\right)$ would generally be an $m-1$ dimensional manifold, and at "almost all" points $y \in \underline{F}\left(P^{*}(x)\right)$, if $\underline{I}_{k}(y) \neq \varnothing$, then $\underline{\underline{I}}(y)$ would have to also contain points $z \neq y$ arbitrarily close to $y$, which precludes Case II from occurring.

Finally, in Case III, neither conditions (i) or (ii) of the theorem and corollary are satisfied vacuously, and here the conditions imply severe restrictions on individual preferences. An illustration of this case is given in Figure 3.2. Again, the frontier of $P^{*}(x)$ coincides with one voter's indifference frontier, in this case that of voter


Figure 3.1-Illustration of Case II.

1. But now, since voter 2's indifference frontier through $y$ also passes through the point $z \in \underline{F}\left(P^{*}(x)\right)$, it follows, by condition (ii), that if voter 2 is not a dummy voter between $y$ and $z$, there must be another voter (in this case voter 4) whose indifference frontier through $y$ also passes through $z$. The same type of coincidence of indifference frontiers must occur for any other voters whose


Figure 3.2-Illustration of Case III.
 Note that the figure illustrates condition (ii) for only one $y \in \underline{F}\left(P^{*}(x)\right)$. It should be kept in mind that condition (ii) implies that similar restrictions must be satisfied for all $y \in \underline{F}\left(P^{*}(x)\right)$. Specifically, because of continuity of preferences (Assumption 1), if $\underline{I F}_{i}(y)$ crosses $\underline{F}\left(P^{*}(x)\right)$ at one point $z \neq y$, then there is a neighborhood $N(y)$ of $y$ such that for any $\hat{y} \in N(y) \cap F\left(P^{*}(x)\right), \underline{I F_{i}}(\hat{y})$ crosses $\underline{F}\left(P^{*}(x)\right)$ at a point $\hat{z} \neq \hat{y}$. Voter $k$ 's indifference frontier must be able to be paired with another voter's indifference frontier for all such $\hat{y}$. In Case III, then, either there must be one "strong" voter with an indifference frontier that coincides with $\underline{F}\left(P^{*}(x)\right)$, and all other voters whose indifference curves cross $\underline{F}\left(P^{*}(x)\right)$ must be dummy voters with respect to any distinct points where they cross, or some voter must have an indifference frontier that crosses $\underline{F}\left(P^{*}(x)\right)$ at two distinct points such that he is not a dummy voter between these points. The first situation, in which there is one strong voter, could arise for general social choice rules. It is obviously precluded by majority rule, and here we would always have the second situation. From the above discussion, we see that this implies severe restrictions on individual preferences, which one would not expect to be met in practice.

Summarizing, we have shown that for $X \subseteq R^{m}$, we would usually expect $\underline{F}\left(P^{*}(x)\right)=\varnothing$ unless there is one "strong voter," $j$, with an indifference contour coinciding with $\underline{F}\left(P^{*}(x)\right)$ or unless all voters' indifference contours cross $\underline{F}\left(P^{*}(x)\right)$ at most once. The latter case can only occur if $m \leqslant 2$, and preferences take a special form over $X$. With these exceptions, we would always expect $\underline{F}\left(P^{*}(x)\right)=\varnothing$ unless extremely strong "symmetry" restrictions on individual utility functions are met at almost all points in $\underline{F}\left(P^{*}(x)\right)$. It follows from the comments at the beginning of this section that if $X$ is connected and $x$ is not a core point, that $\overline{P^{*}(x)}=X$.

The results for general continuous utility functions, then, seem to resemble quite closely those that hold for Euclidian preferences. In general, if $X$ is connected, and $x \in X, \overline{P(x)}=X$, and hence it is possible under the social relation to find a path which begins at $x$ and ends arbitrarily close to any point in the space, even Pareto dominated ones.

Finally, before proceeding, it should be noted that in Theorem 2 and Corollary 1 , the choice of $X$ is arbitrary. Thus, for any $X \subseteq R^{m}$, if $X$ is connected and the Assumptions 1-3 are met in the relative topology on $X$, then we can expect global cycles in $X$ unless the restrictions implied under Case II or III are met. Of course, because of the severity of the conditions of Theorem 2, it is possible that the set $X$ might be chosen in several ways, each of which would lead to global cycles. Thus, by the choice of $X$, there is some control over the path that will be taken from $x$ to $y$. For example, we could force the path to avoid certain alternatives by eliminating them from $X$.

[^2]This section formalizes some of the comments of the previous section by looking at local properties of individual utility functions at points on $\underline{F}\left(P^{*}(x)\right)$. We look only at majority rule, with $n$ odd, in this section, and introduce the additional assumption that utility functions are everywhere differentiable. We then get a restatement of Theorem 2 in terms of individual utility gradients at points on $\underline{F}\left(P^{*}(x)\right)$. It is shown that if $\underline{F}\left(P^{*}(x)\right) \neq \varnothing$, that an extremely strong symmetry condition on individual gradients must be met at all points in $\underline{F}\left(P^{*}(x)\right)$. We call this condition the "joint symmetry" condition. The condition is a natural extension of Plott's conditions for the existence of a core point, and is such that if $X \subseteq R^{m}$, with $m \geqslant 3$, one would not, in general, expect it to be met. It is also shown that this condition is a necessary condition for transitivity of the majority relation in $X$. In other words, for $P$ to be transitive on $X$, the joint symmetry condition must be met at all points $x \in X$. On the other hand, with arbitrary preferences on $R^{m}$, with $m \geqslant 3$, one would expect the condition to be violated almost everywhere in $R^{m}$. It follows that the existence of global intransitivities in $X$ is almost independent of the choice of $X$. From this result it follows that not only will there generally be a majority path between any two points $x$ and $y$, but the path can generally be chosen to be arbitrarily close to any pre-selected curve connecting the two points.

We make the following additional assumption.
Assumption 4: For all $i \in N, u_{i}$ is continuously differentiable on $X$.
We then define the joint symmetry condition.
Definition: The set $A=\left\{\alpha_{1}, \ldots, \alpha_{n}\right\} \subseteq R^{m}$ is said to be jointly symmetric with respect to $j$ if, for all $\alpha_{i} \in A-\left\{\alpha_{j}\right\}, \exists \alpha_{k} \in A-\left\{\alpha_{j}, \alpha_{i}\right\}$ such that $\left\{\alpha_{i}, \alpha_{i}, \alpha_{k}\right\}$ are linearly dependent. The set $A$ is jointly symmetric if for some $j \in N$, it is jointly symmetric with respect to $j$.

The condition of joint symmetry requires, then, that there be some distinguished vector $\alpha_{j} \in A$ such that for all the remaining vectors $\alpha_{i} \in A$ there is at least one additional vector in the space spanned by $\alpha_{i}$ and $\alpha_{j}$. It should be noted that with respect to the usual product topology on $\left(R^{m}\right)^{n}$ for $m \geqslant 3$, the set of points where the joint symmetry condition is violated is an open, dense set in $\left(R^{m}\right)^{n}$. Hence the condition is generically violated if $m \geqslant 3$.

Now for any $y \in X$, we can set

$$
\begin{equation*}
A(y)=\left\{\nabla u_{1}(y), \nabla u_{2}(y), \ldots, \nabla u_{n}(y)\right\} \tag{4.1}
\end{equation*}
$$

So $A(y)$ represents the collection of gradients at any point $y \in X$. We now prove the following extension of Theorem 2 , which proves that if $P^{*}(x)$ has a non-empty frontier, every point in this frontier must satisfy the joint symmetry condition:

Theorem 3: Assume $n$ odd, $R$ is majority rule, all voters satisfy Assumptions 1, 2, 3, and 4, and $X \subset R^{m}$ is open. Then $\exists j \in N$ such that, for all $x \in X$ and $y \in \underline{F}\left(P^{*}(x)\right), A(y)$ is jointly symmetric with respect to $j$.

Proof: By Corollary 2, for some voter $j \in N$ and all $y \in \underline{F}\left(P^{*}(x)\right), \underline{F}\left(P^{*}(x)\right) \subseteq$ $\underline{I F_{j}}(y)$. We will show that for all $y \in \underline{F}\left(P^{*}(x)\right), A(y)$ satisfies the joint symmetry condition with respect to $j$. We assume, for some $y \in \underline{F}\left(P^{*}(x)\right)$, that $A(y)$ does not satisfy joint symmetry with respect to $j$, and derive a contradiction to Corollary 1.

We write $\alpha_{i}=\nabla u_{i}(y)$ for all $i \in N$, so $A(y)=\left\{\alpha_{i} \mid i \in N\right\}$. Now, since joint symmetry is violated at $y$, it follows that for some $i \in N,\left\{\alpha_{i}, \alpha_{i}, \alpha_{k}\right\}$ is linearly independent for all $k \in N-\{i, j\}$. Thus, in particular, $\alpha_{i} \neq 0, \alpha_{i} \neq 0$, and $\alpha_{i} \neq c \alpha_{i}$ for any $c \in R$. So, by Lemmas 7 and 8 , there is a neighborhood, say $N(y) \subseteq X$, of $y$ such that within this neighborhood, $I_{i}(y)$ coincides with $\underline{I F}_{i}(y)$ and $I_{i}(y)$ coincides with $\underline{F}\left(P^{*}(x)\right)$. In other words

$$
\begin{align*}
& N(y) \cap I_{i}(y)=N(y) \cap \underline{I F_{i}}(y),  \tag{4.2}\\
& N(y) \cap I_{j}(y)=N(y) \cap \underline{I F_{j}}(y)=N(y) \cap \underline{F}\left(P^{*}(x)\right) .
\end{align*}
$$

Further, since $u_{i}$ and $u_{i}$ are continuously differentiable, $N(y)$ can be chosen so it also satisfies the condition that for all $w \in N(y)$,

$$
\begin{align*}
& \nabla u_{i}(w) \neq 0,  \tag{4.3}\\
& \nabla u_{j}(w) \neq 0, \\
& \nabla u_{i}(w) \neq c \nabla u_{i}(w) \quad \text { for any } \quad c \in R .
\end{align*}
$$

Then, from (4.2) and Lemma 9, we have, for any $w \in N(y)$,

$$
\begin{align*}
w \in I_{i}(y) \cap I_{i}(y) & \Rightarrow w \in \underline{I F}_{i}(y) \cap \underline{F}\left(P^{*}(x)\right)  \tag{4.4}\\
& \Rightarrow w \in \underline{\hat{F}}_{i}(y) .
\end{align*}
$$

Now, to derive a contradiction to Corollary 2, we must find a point $w \in \underline{F}\left(P^{*}(x)\right)$ such that $w \in \underline{I F}_{i}(y)$ and $w \notin \underline{I F}_{k}(y)$ for any $k \in N-\{j, k\}$. In light of (4.4), and since $\hat{I}_{k}(y) \subseteq \overline{I_{k}}(y)$, it suffices to find a point $w^{*} \in N(y)$ such that

$$
\begin{align*}
& w^{*} \in I_{i}(y) \cap I_{i}(y) \quad \text { and }  \tag{4.5}\\
& w^{*} \notin I_{k}(y) \quad \text { for all } k \in N-\{j, i\} .
\end{align*}
$$

We pick $z \in R^{m}$ such that

$$
\begin{align*}
& z \cdot \alpha_{i}=z \cdot \alpha_{i}=0 \quad \text { and }  \tag{4.6}\\
& z \cdot \alpha_{k} \neq 0
\end{align*}
$$

for all $k \in N-\{j, i\}$. It is clear that we can find such a $z$, because for each $k \in N-\{j, i\}$, since $\left\{\alpha_{j}, \alpha_{i}, \alpha_{k}\right\}$ is linearly independent, we can pick $z_{k} \in R^{m}$ such that $z_{k} \cdot \alpha_{j}=z_{k} \cdot \alpha_{i}=0$, and $z_{k} \cdot \alpha_{k} \neq 0$. (This can be done using the Graham Schmidt procedure, so that $z_{k}$ corresponds to the third member of an orthogonal basis generated by $\left\{\alpha_{i}, \alpha_{i}, \alpha_{k}\right\}$.) Now, we can choose $t_{l} \in R$ for $l \in N-\{j, i\}$ such
that $z=\Sigma_{l \in N-\{j, i\}} t_{l} z_{l}$ satisfies (4.6).
Further, set $\beta_{j}=\alpha_{i} /\left\|\alpha_{j}\right\|$, and $\beta_{i}=\alpha_{i} /\left\|\alpha_{i}\right\|$, and for any $\varepsilon>0$, define

$$
\begin{equation*}
D_{\varepsilon}=\left\{w \mid w=z+a_{i} \beta_{i}+a_{i} \beta_{i}, \text { where }\left|a_{j}\right| \leqslant \varepsilon,\left|a_{i}\right| \leqslant \varepsilon\right\} . \tag{4.7}
\end{equation*}
$$

Now clearly $D_{\varepsilon}$ is compact and convex, for all $\varepsilon>0$. Further since the set of $z$ satisfying $z \cdot \alpha_{k} \neq 0$ is an open set, it follows that we can pick $\varepsilon$ such that for all $w \in D_{\varepsilon}, w \cdot \alpha_{k} \neq 0$ for all $k \in N-\{i, j\}$. Now, by Lemma 6, it follows that for all $k \in N-\{j, i\}$, we can find a $t_{k}^{*} \in R^{+}$such that for $t \leqslant t_{k}^{*}$,

$$
\begin{align*}
& \alpha_{k} \cdot z>0 \Rightarrow y+t D_{\varepsilon} \subseteq P_{k}(y)  \tag{4.8}\\
& \alpha_{k} \cdot z<0 \Rightarrow y+t D_{\varepsilon} \subseteq Q_{k}(y)
\end{align*}
$$

Also, for $\{l, k\}=\{j, i\}$, and $|a| \leqslant \varepsilon$, define

$$
\begin{equation*}
D^{*}(l, a)=\left\{w \in D_{\varepsilon} \mid w=z+a \beta_{l}+a \beta_{k} \text { for some } a_{k} \in R\right\} \tag{4.9}
\end{equation*}
$$

It follows that for $l \in\{j, i\}, D^{*}(l, \varepsilon)$ and $D^{*}(l,-\varepsilon)$ are compact subsets of $D_{\varepsilon}$ satisfying

$$
\begin{array}{lll}
w \cdot \alpha_{l}>0 & \text { for all } & w \in D^{*}(l, \varepsilon)  \tag{4.10}\\
w \cdot \alpha_{l}<0 & \text { for all } & w \in D^{*}(l,-\varepsilon) .
\end{array}
$$

Thus, by Lemma 6 , it follows that we can find a $p^{*} \in R^{+}$such that for $t \leqslant p^{*}$,

$$
\begin{align*}
& y+t D^{*}(l, \varepsilon) \subseteq P_{l}(y)  \tag{4.11}\\
& y+t D^{*}(l,-\varepsilon) \subseteq Q_{l}(y)
\end{align*}
$$

Finally, since $N(y)$ is a neighborhood of $y$, we can pick $q^{*}>0$ such that for $t \leqslant q^{*}$,

$$
\begin{equation*}
w \in y+t D_{\varepsilon} \Rightarrow w \in N(y) . \tag{4.12}
\end{equation*}
$$

## Setting

$$
\begin{equation*}
t^{*}=\min _{k \in N-\{i, j\}}\left(t_{k}^{*}, p^{*}, q^{*}\right) \tag{4.13}
\end{equation*}
$$

it follows that, for $t \leqslant t^{*}$, (4.8), (4.11), and (4.12) are satisfied.
We set $E=y+t^{*} D_{\varepsilon}, E_{l}^{+}=y+t^{*} D^{*}(l, \varepsilon)$, and $E_{l}^{-}=y+t^{*} D(l,-\varepsilon)$. Now, from (4.8) it follows that for all $k \in N-\{j, i\}$, either

$$
\begin{equation*}
E \subseteq P_{k}(y) \quad \text { or } \quad E \subseteq Q_{k}(y) \tag{4.14}
\end{equation*}
$$

Thus, since $P_{k}(y), Q_{k}(y)$, and $I_{k}(y)$ are disjoint, it follows that

$$
\begin{equation*}
E \cap I_{k}(y)=\varnothing \tag{4.15}
\end{equation*}
$$

for all $k \in N-\{j, i\}$. However, we must prove that

$$
\begin{equation*}
E \cap I_{i}(y) \cap I_{i}(y) \neq \varnothing . \tag{4.16}
\end{equation*}
$$

To prove this, note first that, for $l \in\{j, i\}$, it follows from (4.11) that

$$
\begin{align*}
& E_{l}^{+} \subseteq P_{l}(y)  \tag{4.17}\\
& E_{l}^{-} \subseteq Q_{l}(y)
\end{align*}
$$

Now, setting $\varepsilon^{*}=t \varepsilon$, any $w \in E$ can be written in the form

$$
\begin{equation*}
w=\left(y+t^{*} z\right)+b_{i} \boldsymbol{\beta}_{i}+b_{i} \boldsymbol{\beta}_{i} \tag{4.18}
\end{equation*}
$$

where $\left|b_{i}\right| \leqslant \varepsilon^{*}$, and $\left|b_{i}\right| \leqslant \varepsilon^{*}$. So (4.17) can be rewritten as follows. For any $w \in E$, and $l \in\{i, j\}$,

$$
\begin{align*}
& b_{l}=\varepsilon^{*} \Rightarrow u_{l}(w)>u_{l}(y)  \tag{4.19}\\
& b_{l}=-\varepsilon^{*} \Rightarrow u_{l}(w)<u_{l}(y)
\end{align*}
$$

But now, for any $w \in E$, set

$$
\begin{align*}
T(w)=w & +\left(\max \left[\min \left[u_{i}(y)-u_{j}(w), \varepsilon^{*}-b_{i}\right],-\varepsilon^{*}-b_{i}\right]\right) \cdot \beta_{i}  \tag{4.20}\\
& +\left(\max \left[\min \left[u_{i}(y)-u_{i}(w), \varepsilon^{*}-b_{i}\right],-\varepsilon^{*}-b_{i}\right]\right) \cdot \beta_{i} .
\end{align*}
$$

$T: E \rightarrow E$ is a continuous mapping on a compact convex set, and hence by the Brower fixed foint theorem, it follows that there is a fixed point, i.e., a point $w^{*} \in E$ with $T\left(w^{*}\right)=w^{*}$. But from (4.19) and (4.20), it is easily shown that $T\left(w^{*}\right)=w^{*} \Leftrightarrow u_{j}\left(w^{*}\right)=u_{i}(y)$ and $u_{i}\left(w^{*}\right)=u_{i}(y)$. Hence, we have shown existence of a point $w^{*}$ such that

$$
\begin{equation*}
w^{*} \in E \cap I_{i}(y) \cap I_{i}(y) \tag{4.21}
\end{equation*}
$$

and by (4.15), for all $k \in N-\{j, i\}$,

$$
\begin{equation*}
w^{*} \notin I_{k}(y) \tag{4.22}
\end{equation*}
$$

But since $E \subseteq N(y), w^{*}$ satisfies (4.5), and we have a contradiction. Hence, it follows that $A(y)$ must satisfy the joint symmetry condition with respect to $j$, as we wished to show.
Q.E.D.

Theorem 3 can be thought of as a restatement of Corollary 1 in terms of the local conditions that must be satisfied at frontier points of $P^{*}(x)$. As before, if $\underline{F}\left(P^{*}(x)\right)=\varnothing$, then the theorem is satisfied vacuously. However, if $\underline{F}\left(P^{*}(x)\right) \neq \varnothing$, then the joint symmetry condition must be met by the individual utility gradients at all points $y \in \underline{F}\left(P^{*}(x)\right)$. The joint symmetry condition is a condition which, if $m \geqslant 3$, we would rarely, if ever, expect to be met even at just one point in $X$. $A$ fortiori, we would not expect it to be met at all frontier points. Thus we get a further confirmation of the comments of the previous section for $m \geqslant 3$. Namely, if $m \geqslant 3$, and $X \subseteq R^{m}$ is connected, we can virtually always expect, under majority rule, that $\overline{P^{*}(x)}=X$ for any $x \in X$.

It follows further that for the majority relation to be transitive on $X$, we must have the joint symmetry condition met at all points in $X$. This is proven in the following Corollary to Theorem 3. Here, for any $S \subseteq X$, and $x \in X_{0}$, we let $P_{S}^{*}(x)=\bigcup_{j=1}^{\infty} P_{S}^{i}(x)$, where $P_{S}^{1}(x)=\{y \in S \mid y P x\}$, and $P_{S}^{i}(x)=\{y \in S \mid y P z$ for some $\left.z \in P_{S}^{i-1}(x)\right\}$. So $P_{S}^{*}(x)$ is the set of points within $S$ that can be reached by the majority relation.

Corollary 2: If $n$ is odd, $R=f(\bar{R})$ is majority rule, and all voters satisfy Assumptions 1, 2, 3, and 4, with $X$ open, then a necessary condition for $P$ to be transitive on $X$ is that $A(y)$ be jointly symmetric for all $y \in X$. Further, for any $y \in X$, if $A(y)$ is not jointly symmetric, then there is a neighborhood $N(y)$ of $y$ such that, for any $z \in N(y), \overline{P_{N(y)}^{*}(z)}=N(y)$.

Proof: We first prove the second assertion. Assume for some $y \in X, A(y)$ is not jointly symmetric. Then since each $u_{i}$ is continuously differentiable, it follows that we can find a neighborhood $N(y)$ of $y$ (which we may choose to be connected) such that for any $z \in N(y)$, and any $i, j, k \in N$,

$$
\begin{aligned}
& \left\{\nabla u_{i}(y), \nabla u_{i}(y), \nabla u_{k}(y)\right\} \text { linearly independent } \\
& \Rightarrow\left\{\nabla u_{i}(z), \nabla u_{j}(z), \nabla u_{k}(z)\right\} \text { linearly independent. }
\end{aligned}
$$

It follows that, for all $z \in N(y), A(z)$ is not jointly symmetric. But then, by Theorem 3, $\underline{F}\left(P_{N(y)}^{*}(z)\right)=\varnothing$ for all $z \in N(y)$. But further, we cannot have $P_{N(y)}^{*}(z)=\varnothing$ since then, by Plott's Theorem, the joint symmetry condition would be satisfied. Hence, since $P_{N_{(y)}}^{*}(z)$ is open, and $N(y)$ is connected, it follows from Lemma 7 that we must have $\overline{P_{N(y)}^{*}(z)}=N(y)$. Clearly, given any $z \in N(y)$, we can construct a cycle from $y$ to $z$ and back again, so $P$ is not transitive. Q.E.D.

We now consider the implications of the previous theorem and corollary. We assume $X$ is a connected subset of $R^{m}$, where $m \geqslant 3$. We define $\underline{S}(X)$ to be the set of points in $X$ where the joint symmetry condition is satisfied, and let $\underline{V}(X)=$ $X-\underline{S}(X)$ be the set of points in $X$ where the joint symmetry condition is violated. From Assumptions 1-4 it follows that $\boldsymbol{S}(X)$ would generally be a closed set with no interior. $\underline{V}(X)$ would then be an open, dense subset of $X$. Now, from Theorem 1 , it follows that for any $x \in X, \underline{F}\left(P^{*}(x)\right) \subseteq \underline{S}(X)$. But then it follows (see Lemma 5a) that unless $S(X)$, the set of points where joint symmetry is satisfied, chops up $X$ into at least two disjoint open sets, we must have $\underline{F}\left(P^{*}(x)\right)=\varnothing$. Thus, a sufficient condition for $\underline{F}\left(P^{*}(x)\right)=\varnothing$ is that $\underline{V}(X)$ be a connected set.

A further implication of the above arguments is that we may frequently have considerable latitude in choosing a majority path between two points. Thus, for any $X_{0} \subseteq X$, we have $\underline{V}\left(X_{0}\right)=X_{0} \cap \underline{V}(X)$, and $\underline{S}\left(X_{0}\right)=X_{0} \cap \underline{S}(X)$. So if we wish to construct a majority path between $x, y \in X$, we can restrict the path to any subset $X_{0}$ of $X$ such that $x \in X_{0}, y \in X_{0}$ and such that $\underline{V}\left(X_{0}\right)$ is connected. Thus, the path between $x$ and $y$ can be forced to avoid certain alternatives, as in Figure 4.1, by choosing $X_{0}$ appropriately. In fact, it follows further that if $C \subseteq X$ is any simple curve connecting $x$ and $y$ such that $C \cap \underline{S}(X)=\varnothing$, we can construct a "continuous" majority path between $x$ and $y$, each step of which is arbitrarily close to $C$. To see this, note that by Corollary 2, it follows that given any $z \in C$, there is a neighborhood $N(z)$ of $z$, such that $N(z) \subseteq \underline{V}(X)$. Hence, there will be global intransitivities in $N(z)$. Since $C$ is compact, it follows there is a finite subcover $N\left(z_{1}\right), N\left(z_{2}\right), \ldots, N\left(z_{k}\right)$ of $C$, with $N\left(z_{i}\right) \cap N\left(z_{i+1}\right) \neq \varnothing$ for all $i$. There are global intransitivities within each $N\left(z_{i}\right)$, which can be pieced together to form a majority rule path from $x$ to $y$. If each $N(z)$ is chosen so that $N(z) \subseteq$


Figure 4.1-Choice of $X_{0}$ to avoid certain alternatives.


FIGURE 4.2-Construction of majority path arbitrarily close to curve $C$.
$\{y \mid\|z-y\|<\varepsilon\}$, it follows that the resulting path can be forced arbitrarily close to $C$. See Figure 4.2 for an illustration.
Finally, it should be pointed out that even though we would seldom expect the joint symmetry conditions to be met, even if they are met the conditions are very fragile in the same sense that the Plott conditions for an equilibrium are fragile. Namely, when the conditions are met they are vulnerable to misrepresentations or minor perturbations of any one voter's preferences. Specifically, if the joint symmetry conditions are met at a point $y \in X$, and no two voter's gradients are linear combinations of each other, it follows that for any $k \in N$, there is a bogus representation, say $\nabla u_{k}{ }^{*}(y)$ of $\nabla u_{k}(y)$, such that

$$
A_{k}^{*}(y)=\left(A(y)-\left\{\nabla u_{k}(y)\right\}\right) \cup\left\{\nabla u_{k}^{*}(y)\right\}
$$

does not satisfy the joint symmetry conditions. This suggests that results similar to those discussed in McKelvey [13] may hold more generally. Namely that regardless of other voter's preferences, any one voter with complete information of other voter's preferences, control of the agenda, and the ability to cast his own vote as he chooses can always construct majority paths to get anywhere in the space.

In short, we have argued above that if $X \subseteq R^{m}$ is connected, and $m \geqslant 3$, the joint symmetry condition is a severe restriction which at any given point in $X$ we
would expect to fail. We would certainly not expect it to be satisfied on large subsets of $X$. Thus, in general, we would expect $\underline{V}(X)$, the set of points where joint symmetry is violated, to be a connected, dense subset of $X$. But then we have that $\underline{F}\left(P^{*}(x)\right)=\varnothing$ implying that $P^{*}(x)=\varnothing$ or $\overline{P^{*}(x)}=X$. It follows that unless $x$ is a core point, we would generally expect to have naturally occurring majority paths between $x$ and virtually any other point in the space. Even in the rare instances when there are not unlimited majority paths through the entire space, the conditions preventing this state of affairs are fragile enough so that any one voter, through misrepresentation of his own preferences, could give rise to this situation.

## 5. CONCLUSIONS

Despite the impact Arrow's impossibility theorem has had on the study of social choice, there still seems to be a tendency in much of the formal literature dealing with majority rule over multidimensional policy spaces to view majority rule as a fairly well defined notion, which will generally force the social outcome towards "median" like alternatives. Even though it is known that intransitivities will generally exist, a substantial literature has developed on the tacit assumption that these intransitivities are confined to relatively limited areas of the space.

The above results have shown that majority rule is not well defined in the above sense. Rather, the usual situation will be that majority paths exist between any two points in the space. Even in the rare situations when this is not the case, the restrictions implied on individual utility functions seem to be so severe that a minor perturbation of any one voter's preferences would be sufficient to give rise to global intransitivities.

There are several implications of the above results. First, they seem to imply that there are essentially unlimited possibilities for agenda manipulation. Any one voter, with knowledge of other voter's preferences, and the power to set the agenda could, using binary, majority rule based procedures, arrive at any outcome he wants to. See McKelvey [13] for further elaboration on this point. Secondly, these results show the inadequacy of arriving at any useful social choice functions using the notions of top cycle set or the transitive closure of majority rule, as such methods will simply rank all alternatives as socially indifferent. Finally, the results indicate that any attempts to construct positive descriptive theory of political processes based on majority rule (or other social choice functions satisfying the assumptions of this paper) must take account of particular institutional features of these systems, as the social ordering by itself does not give much theoretical leverage. Much work has already been done in this direction, incorporating such institutional features as party competition [6, 11], sequential voting under parliamentary rules $[\mathbf{7}, 9,15]$, structured and unstructured committee environments $[8,16]$, and agenda setters [18].
Carnegie-Mellor University
and
California Institute of Technology
Manuscript received March, 1977; final revision received October, 1978.

## APPENDIX

Here we prove some basic properties of the preference sets for the individual relations $R_{i}$ and the social relation $R=f(\bar{R})$. We first prove a general result, of which we will make frequent use.

LEMMA 1: Let $Y$ be any topological space, and $A \subseteq Y$. If $A \subseteq \bigcup_{i=1}^{n} A_{i}$, where each $A_{i} \subseteq Y$ is closed, with no interior, then $A$ has no interior.

Proof: The proof is by induction on $n$. It is clearly true for $n=1$. Assume, now, that the result is true for $n-1$, but not for $n$. So there is an open set $B$ with

$$
\varnothing \neq B \subseteq A \subseteq\left(\bigcup_{i=1}^{n-1} A_{n}\right) \cup A_{n}
$$

Now if $B \subseteq A_{n}$, then $A_{n}$ has an interior, which is a contradiction. Hence, we set

$$
C=B-A_{n} .
$$

Now $\varnothing \neq C \subseteq\left(\bigcup_{i=1}^{n} A_{i}\right)$, and $C$ is open, which is a contradiction to the induction hypothesis. Hence $A$ can have no interior.
Q.E.D.

We now prove some general properties of the individual preference sets. Note first, that for any complete binary relation $R$ on $X$, that $X$ is partitioned by $P(x), Q(x)$, and $I(x)$. For the individual relations $R_{i}$, under Assumptions 1 and 2 , we have the following further results.

Lemma 2: If Assumptions 1 and 2 are met, for $\bar{R}=\left(R_{1}, \ldots, R_{n}\right) \in \bar{\Theta}$, then for all $i \in N$, and all $x \in X$, (a) $P_{i}(x)$ and $Q_{i}(x)$ are open, and $I_{i}(x)$ is closed with no interior; (b) $\underline{B}\left(P_{i}(x)\right) \cup \underline{B}\left(Q_{i}(x)\right)=I_{i}(x)$; (c) $\underline{F}\left(P_{i}(x)\right)=\underline{F}\left(Q_{i}(x)\right) \subseteq I_{i}(x)$.

Proof: That $P_{i}(x)$ and $Q_{i}(x)$ are open follows directly from Assumption 1, because

$$
\begin{aligned}
& P_{i}(x)=\left\{y \in X \mid u_{i}(y)>u_{i}(x)\right\}=u_{i}^{-1}\left(\left\{t \in R \mid t>u_{i}(x)\right\}\right), \\
& Q_{i}(x)=\left\{y \in X \mid u_{i}(x)>u_{i}(y)\right\}=u_{i}^{-1}\left(\left\{t \in R \mid t<u_{i}(x)\right\}\right),
\end{aligned}
$$

but since $u_{i}$ is continuous, the inverse image of every open set is open, hence $P_{i}(x)$ and $Q_{i}(x)$ are both open. Next, since

$$
I_{i}(x)=X-P_{i}(x)-Q_{i}(x)
$$

it follows that $I_{i}(x)$ is closed. By Assumption 2, $I_{i}(x)$ has no interior.
Now, to prove (b), since $I_{i}(x)$ has no interior, it follows that any point $I_{i}(x)$ is an accumulation point of $P_{i}(x)$ or $Q_{i}(x)$, i.e.,

$$
I_{i}(x) \subseteq \underline{B}\left(P_{i}(x)\right) \cup \underline{B}\left(Q_{i}(x)\right) .
$$

But further, since $P_{i}(x)$ and $Q_{i}(x)$ are both open, with $\overline{P_{i}(x)} \subseteq\left(Q_{i}(x)\right)^{c}$ and $\overline{Q_{i}(x)} \subseteq\left(P_{i}(x)\right)^{c}$, it follows that $\underline{B}\left(P_{i}(x)\right) \subseteq I_{i}(x)$ and $\underline{B}\left(Q_{i}(x)\right) \subseteq I_{i}(x)$, so $\underline{B}\left(P_{i}(x)\right) \cup \underline{B}\left(Q_{i}(x)\right) \subseteq I_{i}(x)$. Hence

$$
I_{i}(x)=\underline{B}\left(P_{i}(x)\right) \cup \underline{B}\left(Q_{i}(x)\right) .
$$

To prove (c), we first prove that

$$
\overline{P_{i}(x)}=\overline{\left[P_{i}(x) \cup I_{i}(x)\right]^{0}}
$$

and

$$
\overline{Q_{i}(x)}=\overline{\left[Q_{i}(x) \cup I_{i}(x)\right]^{0}} .
$$

First, to prove $\overline{P_{i}(x)} \subseteq \overline{\left[P_{i}(x) \cup I_{i}(x)\right]^{0}}$, pick $y \in \overline{P_{i}(x)}$, and let $N(y)$ be an arbitrary neighborhood of $y$.

Then since $P_{i}(x)$ is open, $\quad P_{i}(x)=\left[P_{i}(x)\right]^{0}$, so $N(y) \cap P_{i}(x) \neq \varnothing \Rightarrow N(y) \cap P_{i}(x)^{0} \neq \varnothing \Rightarrow$ $N(y) \cap\left[P_{i}(x) \cup I_{i}(x)\right]^{0} \neq \varnothing \Rightarrow y \in\left[\overline{\left.P_{i}(x) \cup I_{i}(x)\right]^{0}}\right.$. To show $\overline{\left[P_{i}(x) \cup I_{i}(x)\right]^{0}} \subseteq P_{i}(x)$, pick $y \in$ $\overline{\left[P_{i}(x) \cup I_{i}(x)\right]^{0}}$, and let $N(y)$ be an open neighborhood of $y$. Then $N(y) \cap\left[P_{i}(x) \cup I_{i}(x)\right]^{0} \neq \varnothing$. So pick $z \in N(y) \cap\left[P_{i}(x) \cup I_{i}(x)\right]^{0}$. Since both $N(y)$ and $\left[P_{i}(x) \cup I_{i}(x)\right]^{0}$ are open, we can find an open neighborhood $N(z)$ of $z$ with $N(z) \subseteq N(y)$ and $N(z) \subseteq\left[P_{i}(x) \cup I_{i}(x)\right]^{0} \Rightarrow N(z) \subseteq P_{i}(x) \cup I_{i}(x)$. But then we must have $N(z) \cap P_{i}(x) \neq \varnothing$, otherwise $N(z) \subseteq I_{i}(x)$, a contradiction to Assumption 2, since $I_{i}(x)$ has no interior. But since $N(z) \subseteq N(y)$, it follows that $N(y) \cap P_{i}(x) \neq \varnothing \Rightarrow y \in \overline{P_{i}(x)}$, as we wished to show. Thus we have proven that $\overline{P_{i}(x)}=\overline{\left[P_{i}(x) \cup I_{i}(x)\right]^{0}}$. The proof that $\overline{Q_{i}(x)}=\overline{\left[Q_{i}(x) \cup I_{i}(x)\right]}$ is exactly equivalent.

Now to show $\underline{F}\left(P_{i}(x)\right)=\underline{F}\left(Q_{i}(x)\right)$, we have, by the definition of the frontier of a set,

$$
\begin{aligned}
\underline{F}\left(P_{i}(x)\right) & =\overline{\left(P_{i}(x)\right)^{0}} \cap \overline{\left(\left(P_{i}(x)\right)^{c}\right)^{0}} \\
& =\overline{P_{i}(x)} \cap \overline{\left(Q_{i}(x) \cup I_{i}(x)\right)^{0}} \\
& =\overline{\left[P_{i}(x) \cup I_{i}(x)\right]^{0}} \cap \overline{Q_{i}(x)} \\
& =\overline{\left(\left(Q_{i}(x)\right)^{c}\right)^{0}} \cap \overline{\left(Q_{i}(x)\right)^{0}} \\
& =\underline{F}\left(Q_{i}(x)\right) .
\end{aligned}
$$

Finally, since $\underline{F}\left(P_{i}(x)\right) \subseteq \underline{B}\left(P_{i}(x)\right)$, it follows from (b) that $\underline{F}\left(P_{i}(x)\right)=\underline{F}\left(Q_{i}(x)\right) \subseteq I_{i}(x) . \quad$ Q.E.D.
The next result proves that the properties of individual preference sets given in Lemma 1 are inherited by the social relation. Also some relations between the individual and social preference sets are proven.

Lemma 3: If $R=f(\bar{R})$, where $f \in \mathscr{F}$, and Assumptions 1 and 2 are met, then for any $x \in X$, (a) $P(x)$
 $\underline{F}(Q(x)) \subseteq I(x) ;(\mathrm{d}) I(x) \subseteq \bigcup_{i \in N} I_{i}(x) ;$ (e) $\underline{F}(P(x)) \subseteq \bigcup_{i \in N} \underline{F}\left(P_{i}(x)\right)$.

Proof: We prove (d) first. To prove (d), let $y \in I(x)$, and assume $y \notin \bigcup_{i \in N} I_{i}(x)$. Then let $C_{1}=\left\{i \in N \mid y \in P_{i}(x)\right\}, C_{2}=\left\{i \in N \mid y \in Q_{i}(x)\right\}$, and $C_{3}=\left\{i \in N \mid y \in I_{i}(x)\right\}$. By Lemma 2a, $C_{1}, C_{2}$, and $C_{3}$ partition $N$, since any $i \in N$ must be a member of one of the three sets. Further, since $y \notin \bigcup_{i=1}^{n} I_{i}(x)$, it follows that $C_{3}=\varnothing$. Thus $C_{1}=N-C_{2}$. Therefore, by (2.1b), it follows that either $C_{1} \in \underline{W}$ or $C_{2} \in \underline{W}$. But, by construction $y P_{C_{1}} x$ and $x P_{C_{2}} y$. Hence, by (2.2), $C_{1} \in \underline{W} \Rightarrow y P x$ and $C_{2} \in \underline{W} \Rightarrow x P y$. So in either case, $y \notin I(x)$, a contradiction. Hence $y \in \bigcup_{i=1}^{n} I_{i}(x)$, and it follows that $I(x) \subseteq \bigcup_{i=1}^{n} I_{i}(x)$.

We next prove (a). To see that $P(x)$ and $Q(x)$ are open, we note that since $f \in \mathscr{F}$, we can write $P(x)$ and $Q(x)$ as follows:

$$
P(x)=\bigcup_{C \in \underline{W}} \bigcap_{i \in C} P_{i}(x)
$$

and

$$
Q(x)=\bigcup_{C \in W} \bigcap_{i \in C} Q_{i}(x) .
$$

Since finite unions and intersections of open sets are open, it follows that $P(x)$ and $Q(x)$ are open. Since $I(x)=X-P(x)-Q(x), I(x)$ is closed. That $I(x)$ has no interior follows directly from Lemma 1 and Lemma 3d. From Lemma 3d it follows that $I(x) \subseteq \bigcup_{i \in N} I_{i}(x)$, where each $I_{i}(x)$ is closed with no interior. Hence by Lemma $1, I(x)$ has no interior.

The proofs of (b) and (c) follow exactly the pattern of the same proofs in Lemma 2, as the only properties that were needed in that proof were that $P_{i}(x)$ and $Q_{i}(x)$ were open, and $I_{i}(x)$ was closed with no interior. Since $I(x), P(x)$, and $Q(x)$ satisfy these same properties, the proofs go through unchanged.

Finally, to show (e), we again argue by contradiction, and assume $y \in \underline{F}(P(x))$, with $y \notin \bigcup_{i \in N} \underline{F}\left(P_{i}(x)\right)$. It follows that we can find an open set $B$, with $y \in B$ such that for all $i \in N$, either $B \cap P_{i}(x)=\varnothing$ or $B \cap Q_{i}(x)=\varnothing$. We set $C_{1}=\left\{i \in N \mid B \cap P_{i}(x)=\varnothing\right\}$, and $C_{2}=\left\{i \in N \mid B \cap Q_{i}(x)=\varnothing\right\}$.

Then $C_{1}$ and $C_{2}$ partition $N$, hence by (2.1b), either $C_{1} \in \underline{W}$ and $C_{2} \notin W$, or $C_{1} \notin \underline{W}$ and $C_{2} \in \underline{W}$. In the first case, by (2.1a), it follows that for any $C \subseteq C_{2}, C \notin W$. But for any $z \in B,\left\{i \in N \mid z P_{i} x\right\} \subseteq C_{2}$. It follows (by (2.2)) that we cannot have $z P x$. In other words, $z \notin P(x)$, for any $z \in B$, or $B \cap P(x)=\varnothing \Rightarrow$ $y \notin \underline{F}(P(x))$. Since $\underline{F}(P(x))=\underline{F}(Q(x))$ (from Lemma 3(c)) it foliows that $y \notin \underline{F}(P(x))$ in the second case also. Thus, we have a contradiction, and hence we must have $y \in \bigcup_{i \in N} \underline{F}\left(P_{i}(x)\right)$. Thus $\underline{F}(P(x)) \subseteq$ $\bigcup_{i \in N} \underset{\underline{F}}{ }\left(P_{i}(x)\right)$, and (e) is proven.
Q.E.D.

The fourth lemma proves some properties of the higher order preference sets $P^{i}(x)$ and $P^{*}(x)$ for the social relation. We need a definition first. If $\Gamma: X \rightarrow 2^{X}$ is a correspondence on $X$, we say $\Gamma$ is lower semi-continuous if for every $x_{0} \in X$, and every open set $G \subseteq X$ with $G \cap \Gamma\left(x_{0}\right) \neq \varnothing$, there is a neighborhood $N\left(x_{0}\right)$ of $x_{0}$ such that $x \in N\left(x_{0}\right) \Rightarrow \Gamma(x) \cap G \neq \varnothing$. We now show that $P^{i}(x)$ is lower semi-continuous for all $j$, as is $P^{*}(x)$.

Lemma 4: If $R=f(\bar{R})$, where $f \in \mathscr{F}$, and Assumption 1 is met, then (a) for all $x \in X$ and $j \geqslant 1, P^{i}(x)$ and $P^{*}(x)$ are open sets; (b) for all $j \geqslant 1, P^{i}(x)$ is lower semi continuous, as is $P^{*}(x)$.

Proof: Lemma 3(a) proves that $P(x)=P^{1}(x)$ is open. It should be noted that this proof depends only on Assumption 1. To prove that $P^{i}(x)$ is open, note that

$$
P^{i}(x)=\bigcup_{y \in P^{i-1}(x)} P^{1}(y) .
$$

Since $P^{i}(x)$ is an infinite union of open sets, it is open. Next,

$$
P^{*}(x)=\bigcup_{j=1}^{\infty} P^{i}(x)
$$

So $P^{*}(x)$ is also an infinite union of open sets and is open, and (a) is proven.
Now, to prove (b), we first prove $P(x)=P^{1}(x)$ is lower semi-continuous. So let $x_{0} \in X$, and let $G \subseteq X$ be open with $G \cap P^{1}\left(x_{0}\right) \neq \varnothing$. We must show there is a neighborhood $N\left(x_{0}\right)$ of $x_{0}$ such that $x \in N\left(x_{0}\right) \Rightarrow P^{1}(x) \cap G \neq \varnothing$. So let $y \in G \cap P^{1}\left(x_{0}\right)$, and set

$$
N\left(x_{0}\right)=\left\{x \in X \mid y \in P^{1}(x)\right\}=Q^{1}(y)
$$

From Lemma 2, $N\left(x_{0}\right)$ is open. Also $N\left(x_{0}\right)$ clearly contains $x_{0}$. So $N\left(x_{0}\right)$ is a neighborhood of $x_{0}$. Also, for all $x \in N\left(x_{0}\right), y \in P^{1}(x)$. Hence $P^{1}(x) \cap G \neq \varnothing$, so $P^{1}(x)$ is lower semi-continuous.

To prove that $P^{i}(x)$ is lower semi-continuous, it suffices to note that $P^{i}$ is the composition product of $P^{i-1}$ and $P^{1}$. In other words, $P^{i}(x)=P^{1} \circ P^{i-1}(x)$. Since the composition product of two lower semi-continuous correspondences is lower semi-continuous [ $\mathbf{2}$, Theorem 1, p. 113], it follows by a simple induction argument that $P^{i}(x)$ is lower semi-continuous, for all $j$.

Now $P^{*}(x)=\bigcup_{i=1}^{\infty} P^{i}(x)$ is the union of a family of lower semi-continuous mappings. It follows [2, Theorem 2, p. 114] that $P^{*}(x)$ is lower semi-continuous.
Q.E.D.

Lemma 5: If $X$ is any connected topological space, and $S \subseteq X$ is open, then (a) $X-\underline{F}(S)$ connected $\Rightarrow$ $\underline{F}(S)=\varnothing$; (b) $\underline{F}(S)=\varnothing \Rightarrow S=\varnothing$ or $\bar{S}=X$.

Proof: To prove (a), assume $X-\underline{F}(\underline{S})$ is connected and that $\underline{F(S)} \neq \varnothing$, say $y \in \underline{F}(S)$. Then by definition of $\underline{F}(S)$, it follows that $y \in \overline{S^{0}}$ and $y \in \overline{\left(S^{c}\right)^{0}}$. Let $A=\overline{S^{0}}, B=\overline{\left(S^{c}\right)^{0}}$, and $C=\underline{F}(S)$; then it is easily verified that $A^{0}, B^{0}$, and $C$ partition $X$. Hence $A^{0}$ and $B^{0}$ partition $X-C=X-\underline{F}(S)$. Further, both $A^{0}$ and $B^{0}$ are non-empty, since $y \in A, y \in B$, and both $A$ and $B$ are the closure of open sets. But then $X-\underline{F}(S)$ is not connected, in contradiction to the assumption of the Lemma. Hence we must have $\underline{F}(S)=\varnothing$.

Now, to prove (b), by the definition of $\underline{F}(S), \underline{F}(S)=\varnothing \Leftrightarrow$

$$
\begin{equation*}
\overline{S^{0}} \cap \overline{\left(S^{c}\right)^{0}}=\varnothing \tag{A.1}
\end{equation*}
$$

But now if $S$ is open, then

$$
\begin{equation*}
\overline{S^{0}} \cup \overline{\left(S^{c}\right)^{0}}=X \tag{A.2}
\end{equation*}
$$

because if $x \notin \overline{S^{0}}$, then there is a neighborhood $N(x)$ of $x$ such that

$$
\begin{aligned}
N(x) \cap S^{0}=\varnothing & \Rightarrow N(x) \cap S=\varnothing \Rightarrow x \in\left(S^{c}\right)^{0} \\
& \Rightarrow x \in\left(\overline{\left.S^{c}\right)^{0}} .\right.
\end{aligned}
$$

But now since $X$ is connected, it cannot be expressed as a disjoint union of two nonempty closed sets, hence (A2) implies that $\overline{S^{0}}=\varnothing$ or $\overline{\left(S^{c}\right)^{0}}=\varnothing$. But since $S$ is open, $\overline{S^{0}}=\varnothing \Rightarrow \bar{S}=\varnothing \Rightarrow S=\varnothing$. On the other hand $\overline{\left(S^{c}\right)^{0}}=\varnothing \Rightarrow\left(S^{c}\right)^{0}=\varnothing$. But then for any $x \in S^{c}$, and every neighborhood $N(x)$ of $x$, $N(x) \cap\left(S^{c}\right)^{c} \neq \varnothing \Rightarrow N(x) \cap S \neq \varnothing \Rightarrow x \in \bar{S}$. Since $x \in S \Rightarrow x \in \bar{S}$, it follows that for all $x \in X, x \in \bar{S}$. In other words, $\bar{S}=X$. Thus we have shown that either $S=\varnothing$ or $\bar{S}=X$, as we wished to show. $\quad$ Q.E.D.

Lemma 6: Let $u_{i}: X \rightarrow R$ be continuously differentiable on $X \subseteq R^{m}$, and let $y \in X^{0}$. Then if $A \subseteq R^{m}$ is compact, and $z \cdot \nabla u_{i}(y)>0$ for all $z \in A$ then $\exists t^{*} \in R$, with $t^{*}>0$, such that for all $0<t \leqslant t^{*}, y+t A \subseteq$ $P_{i}(y)$. Similarly if $z \cdot \nabla u_{i}(y)<0$ for all $z \in A$, then $\exists t^{*} \in R^{+}$such that for all $0<t \leqslant t^{*}, y+t A \subseteq Q_{i}(y)$.

Proof: For any $z \in A, \exists t_{z} \in R$, with $t_{z}>0$, such that $y+t z \in X$ and $u_{i}(y+t z)>u_{i}(y)$ for all $t \leqslant t_{z}$. (See, e.g., Zangwill [25, Theorem 2.1, p. 24] for a proof of this.) In other words, $y+t z \in P_{i}(y)$. But since $P_{i}(y)$ is open in the relative topology on $X$ (by Lemma 2(a)), it follows that there is an open neighborhood, $N(z)$, of $z$ such that $y+t(N(z)) \subseteq P_{i}(y)$ whenever $t<t_{z}$. Now, by the Axiom of Choice, for any $z \in A$, we can find an open neighborhood $N(z)$ of $z$, and a $t_{z} \in R$ with $t_{z}>0$ such that $y+t(N(z)) \subseteq P_{i}(y)$ for $t \leqslant t_{z}$. Now $\{N(z) \mid z \in A\}$ is an open covering of $A$, hence, since $A$ is compact, by the Heine-Borel Theorem it follows that there is a finite subcovering, say $\left\{N\left(z_{1}\right), \ldots, N\left(z_{K}\right)\right\}$. Now setting $t^{*}=\min _{1 \leqslant i \leqslant K} t_{z i}$, it follows that if $t \leqslant t^{*}$, for any $z \in A$, then $z \in N\left(z_{i}\right)$ for some $1 \leqslant i \leqslant K$. In other words, $y+t_{z} \subseteq P_{i}(y)$ since $t \leqslant y^{*} \leqslant t_{z i}$. But, then, we have just shown that $y+t A \subseteq P_{i}(y)$ whenever $t<t^{*}$, as we wished to show.
The proof of the second assertion of the theorem is exactly analogous to the above proof.
Q.E.D.

Lemma 7: Let $X \subseteq R^{m}, u_{i}: X \rightarrow R$ be continuously differentiable on $X$, and let $y \in X^{0}$ satisfy $\nabla u_{i}(y) \neq 0$. Then there is a neighborhood $N(y)$ of $y$ such that $N(y) \cap I_{i}(y)=N(y) \cap \underline{I F}_{i}(y)$.

Proof: Since $u_{i}$ is continuously differentiable, and $\nabla u_{i}(y) \neq 0$, we can find a neighborhood $N(y)$ of $y$ such that $\nabla u_{i}(w) \neq 0$ for all $w \in N(y)$. Now let $w \in N(y) \cap I_{i}(y)$. We let $\alpha_{i}=\nabla u_{i}(w)$, and set $B=\left\{z \in R^{m} \mid\|z\| \leqslant\left\|\alpha_{i}\right\| / 2\right\}$. Then we set

$$
\begin{aligned}
& D^{+}=\alpha_{i}+B \\
& D^{-}=-\alpha_{i}+B .
\end{aligned}
$$

It follows that $z \cdot \alpha_{i}>0$ for all $z \in D^{+}$, and $z \cdot \alpha_{i}<0$ for all $z \in D^{-}$. Further $D^{+}$and $D^{-}$are compact. Thus, by Lemma $6, \exists t^{*}$ such that for $t<t^{*}, w+t D^{+} \subseteq P_{i}(w)=P_{i}(y)$, and $w+t D^{-} \subseteq Q_{i}(w)=Q_{i}(y)$. Hence, $w \in \underline{I F}(y)$, and we have shown $N(y) \cap I_{i}(y) \subseteq N(y) \cap \underline{I F}_{i}(y)$. The reverse inclusion is trivial.
Q.E.D.

Lemma 8: Let Assumptions $1-4$ hold for all $i \in N$, let $x \in X$, and let $y \in X^{0}$ satisfy $y \in \underline{F}\left(P^{*}(x)\right)$. Then if $\nabla u_{i}(y) \neq 0$ for all $i \in N$, there is a neighborhood, $N(y)$ of $y$ such that for some $j \in N$,

$$
N(y) \cap \underline{F}\left(P^{*}(x)\right)=N(y) \cap I_{i}(y)=N(y) \cap \underline{I F}_{i}(y) .
$$

Proof: By Theorem 2, for some $j \in N, \underline{F}\left(P^{*}(x)\right) \subseteq I_{i}(y) \subseteq \underline{I F}_{i}(y)$. By Lemma 7 there is an open neighborhood, $N(y)$ of $y$ such that $N(y) \cap I_{j}(y)=N(y) \cap \underline{I F_{j}}(y)$. Further, since $\nabla u_{j}(y) \neq 0$, and $u_{j}$ is continuously differentiable, we can pick $N(y)$ so that

$$
\begin{aligned}
& A=N(y) \cap P_{j}(y) \quad \text { is connected and } \\
& B=N(y) \cap Q_{i}(y) \quad \text { is connected. }
\end{aligned}
$$

Further, since $y \in I F_{j}(y)$, both $A$ and $B$ are non-empty. Now, from Theorem 2 and Lemma 7, we have

$$
N(y) \cap \underline{F}\left(P^{*}(x)\right) \subseteq N(y) \cap I_{j}(y)=N(y) \cap \underline{I F_{j}}(y) ;
$$

we must only show

$$
\begin{equation*}
N(y) \cap \underline{I F}_{j}(y) \subseteq N(y) \cap \underline{F}\left(P^{*}(x)\right) . \tag{A.3}
\end{equation*}
$$

Suppose, for some $z \in N(y), z \in \underline{I F_{j}}(y)$ and $z \notin \underline{F}\left(P^{*}(x)\right)$. Then it follows that
(A.4) $\quad N(y)-\underline{F}\left(P^{*}(x)\right)$ is connected.

To see this, note that if $N(y)-\underline{F}\left(P^{*}(x)\right)$ is not connected, then we can find two disjoint open sets, say $C$, $D \subseteq X$ such that

$$
\begin{aligned}
& N(y)-\underline{F}\left(P^{*}(x)\right)=C \cup D, \\
& {\left[N(y)-\underline{F}\left(P^{*}(x)\right)\right] \cap C \neq \varnothing,}
\end{aligned}
$$

and

$$
\left[N(y)-\underline{F}\left(P^{*}(x)\right)\right] \cap D \neq \varnothing .
$$

But then $z \in C$ or $z \in D$. Assume without loss of generality that $z \in C$. Then since $z \in I F_{j}(y)$, and $C$ is open, it follows that $C \cap A \neq \varnothing$ and $C \cap B \neq \varnothing$. Also, it follows that either $D \cap A \neq \varnothing$ or $D \cap B \neq \varnothing$. Assume without loss of generality that $D \cap A \neq \varnothing$. Then

$$
\begin{aligned}
& C \cap A \neq \varnothing, \\
& D \cap A \neq \varnothing,
\end{aligned}
$$

and

$$
A \subseteq C \cup D
$$

Thus $A$ is not connected, a contradiction. Thus we have established $N(y)-\underline{F}\left(P^{*}(x)\right)$ is connected. But now by Lemma $5(\mathrm{a}), N(y)-\underline{F}\left(P^{*}(x)\right)$ connected $\Rightarrow N(y) \cap \underline{F}\left(P^{*}(x)\right)=\varnothing$, which contradicts the assumption that $y \in \underline{F}\left(P^{*}(x)\right)$. Hence (A.3) is established, which proves the result.

Lemma 9: Let Assumptions 1-4 hold for all $i \in N$, let $x \in X, y \in X^{0}$ satisfy $y \in \underline{F}\left(P^{*}(x)\right)$, and assume $\underline{F}\left(P^{*}(x)\right) \subseteq I_{j}(y)$. Then if $\nabla u_{j}(y) \neq 0$ and for some $i \in N-\{j\}, \nabla u_{i}(y) \neq c \nabla u_{j}(y)$ for all $c \in R$, there is a neighborhood $N(y)$ of $y$ such that, for all $w \in N(y)$,

$$
w \in I_{i}(y) \cap I_{i}(y) \Rightarrow w \in I \hat{F}_{i}(y) .
$$

Proof: We pick $N(y)$ to satisfy $\nabla u_{i}(w) \neq c \nabla u_{i}(w)$ for all $w \in N(y)$, and to simultaneously satisfy the conditions that $\nabla u_{j}(w) \neq 0, \nabla u_{i}(w) \neq 0$ for all $w \in N(y)$. Then by Lemmas 7 and 8 it follows that

$$
N(y) \cap I_{i}(y)=N(y) \cap \underline{I F}_{i}(y)
$$

and

$$
N(y) \cap I_{j}(y)=N(y) \cap \underline{I} \underline{F}_{j}(y)=N(y) \cap \underline{F}\left(P^{*}(x)\right) .
$$

Thus, for any $w \in N(y)$,

$$
\begin{aligned}
w \in I_{i}(y) \cap I_{i}(y) & \Rightarrow w \in \underline{I F}_{i}(y) \cap \underline{I F}_{i}(y) \\
& \Rightarrow w \in \underline{I F}_{i}(w) \cap \underline{F}_{i}(w) \\
& \Rightarrow w \in \underline{I}_{i}(w) \cap \underline{F}\left(P^{*}(x)\right) .
\end{aligned}
$$

We need only show that in any neighborhood of $w$ there are points in $\underline{F}\left(P^{*}(x)\right)$ which are preferred by $i$ to $w$ and points to which he prefers $w$. Pick $z \in R^{m}$ such that $z \cdot \nabla u_{i}(w)=0$ and $z \cdot \nabla u_{i}(w)>0$. Let $N(z)$ be a closed neighborhood of $z$ such that for all $z^{*} \in N(z), z^{*} \cdot \nabla u_{i}(w)>0$. Then by Lemma 6 ,
there is a $t^{*}$ such that, for $0<t<t^{*}$,

$$
\begin{align*}
& \varnothing \neq(w+t N(z)) \cap I_{i}(y) \subseteq P_{i}(w) \cap I_{i}(y)=P_{i}(w) \cap \underline{F}\left(P^{*}(x)\right), \\
& \varnothing \neq(w-t N(z)) \cap I_{i}(y) \subseteq Q_{i}(w) \cap I_{i}(y)=Q_{i}(w) \cap \underline{F}\left(P^{*}(x)\right) .
\end{align*}
$$

## REFERENCES

[1] Arrow, K. J.: Social Choice and Individual Values (2nd ed.). New Haven: Yale University Press, 1963.
[2] Berge, C.: Topological Spaces (translated by E. M. Patterson). New York: Macmillan, 1963.
[3] Cohen, L.: "Cyclic Sets in Multidimensional Voting Models," Journal of Economic Theory (forthcoming).
[4] Cohen, L., and S. Matthews: "Constrained Plott Equilibria, Directional Equilibria and Global Cycling Sets," Review of Economic Studies (forthcoming).
[5] Davis, O. A., M. H. Degroot, and M. J. Hinich: "Social Preference Orderings and Majority Rule," Econometrica, 40 (1972), 147-157.
[6] Davis, O. A., M. J. Hinich, and P. C. Ordeshook: "An Expository Development of a Mathematical Model of the Electoral Process," American Political Science Review, 64 (1970), 426-448.
[7] Farquharson, R.: Theory of Voting. New Haven: Yale University Press, 1969.
[8] Ferejohn, J., M. Fiorina, and E. Packel: "A Non Equilibrium Approach to Legislative Decision Theory," Social Science Working Paper No. 202, California Institute of Technology, 1978.
[9] Kramer, G. H.: "Sophisticated Voting Over Multidimensional Choice Spaces," Journal of Mathematical Sociology, 2 (1972), 165-180.
[10] -: "On a Class of Equilibrium Conditions for Majority Rule," Econometrica, 41 (1973), 285-297.
[11] ——: "A Dynamical Model of Equilibrium," Cowles Foundation, Yale University, Discussion Paper No. 36, 1975.
[12] Matthews, S.: "The Possibility of Voting Equilibria," Mimeo, California Institute of Technology, Division of Humanities and Social Sciences, 1977.
[13] MCKelvey, R. D.: "Intransitivities in Multidimensional Voting Models and Some Implications for Agenda Control," Journal of Economic Theory, 12 (1976), 472-482.
[14] McKelvey, R. D., and R. E. Wendell: "Voting Equilibria in Multidimensional Choice Spaces," Mathematics of Operations Research, 1 (1976), 144-158.
[15] McKelvey, R. D., and R. G. Niemi: "A Multistage Game Representation of Sophisticated Voting for Binary Procedures," Journal of Economic Theory, 18 (1978), 1-22.
[16] McKelvey, R. D., P. C. Ordeshook, and M. Winer: "The Competitive Solution for n-Person Games Without Side Payments," American Political Science Review, 72 (1978), 599-615.
[17] Plott, C. R.: "A Notion of Equilibrium and Its Possibility Under Majority Rule," American Economic Review, 57 (1967), 787-806.
[18] Plott, C. R., and M. E. Levine: "A Model of Agenda Influence on Committee Decisions," American Economic Review, 68 (1978), 146-160.
[19] Rubinstein, A.: "A Note About the 'Nowhere Denseness' of Societies Having an Equilibrium Under Majority Rule," Econometrica (forthcoming).
[20] Schofield, N.: "Instability of Simple Dynamic Games," Review of Economic Studies, 40 (1978), 575-594.
[21] -: "Generic Instability of Voting Games," Mimeo, Department of Government, University of Texas at Austin, 1978.
[22] Sen, A. K.: "Social Choice Theory: A Re-examination," Econometrica, 45 (1977), 53-90.
[23] Sloss, J.: "Stable Points of Directional Preference Relations," Technical Report No. 71-7, Operations Research House, Stanford University, 1971.
[24] Slutsky, S.: "Equilibrium Under $\alpha$ Majority Rule," Mimeo. Paper presented at the 1978 Meetings of the Public Choice Convention, New Orleans, Louisiana.
[25] Zangwill, W. I.: Nonlinear Programming, A Unified Approach. Englewood Cliffs, N.J.: Prentice-Hall, 1969.


[^0]:    ${ }^{1}$ This research was supported, in part, by the National Science Foundation, Grant \#SOC7708291. I am indebted to Norman Schofield for some conversations in the early stages of this research which influenced my thinking on the problem, and to Rodney Gretlein for comments on an earlier draft. In addition to the literature cited in the text, the interested reader should also see recent articles by Cohen and Matthews [4] and Schofield [21], which were written subsequent to this article, and extend some of the results of Section 4 of this paper.

[^1]:    ${ }^{2}$ By the definition of continuity, the inverse image of every open set is open. Hence, since $X$ is finite, we can find a small enough open neighborhood $B \subseteq R$ around $u_{i}(y)$ such that

    $$
    u_{i}^{-1}(B)=I_{i}(y) .
    $$

[^2]:    ${ }^{3}$ The theorem and corollary do not imply that the indifference frontiers must be paired in a $1-1$ fashion, as drawn in the illustration (i.e., they do not prevent three voter's indifference frontiers from passing through the point $z$ ). However, for majority rule it can be shown that the above pairing is, in fact, 1-1. Further it can be shown that given any voter $i \in N-\{j\}, \underline{F}\left(P^{*}(x)\right)$ can be partitioned into sets such that within each set, there is a voter $k \in N-\{j, i\}$ whose preferences on $\underline{F}\left(P^{*}(x)\right)$ are essentially opposite to those of voter $i$. A future paper will report on this.

