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OTHER SOLUTIONS TO NASH'S BARGAINING PROBLEM

BY EHUD KALAI AND MEIR SMORODINSKY

A two-person bargaining problem is considered. It is shown that under four axioms that describe the behavior of players there is a unique solution to such a problem. The axioms and the solution presented are different from those suggested by Nash. Also, families of solutions which satisfy a more limited set of axioms and which are continuous are discussed.

1. INTRODUCTION

WE CONSIDER a two-person bargaining problem mathematically formulated as follows. To every two-person game we associate a pair (a, S), where a is a point in the plane and S is a subset of the plane. The pair (a, S) has the following intuitive interpretation: $a = (a_1, a_2)$ where a_i is the level of utility that player i receives if the two players do not cooperate with each other. Every point $x = (x_1, x_2) \in S$ represents levels of utility for players 1 and 2 that can be reached by an outcome of the game which is feasible for the two players when they do cooperate. We are interested in finding an outcome in S which will be agreeable to both players.

This problem was considered by Nash [**3**] and his classical result was that under certain axioms there is a unique solution. However, one of his axioms of independence of irrelevant alternatives came under criticism (see [**2**, p. 128]). In this paper we suggest an alternative axiom which leads to another unique solution. Also, it was called to our attention by the referee that experiments conducted by H. W. Crott [**1**] led to the solution implied by our axioms rather than to Nash's solution.

We also consider the class of continuous solutions which are required to satisfy only the axioms of Nash which are usually accepted. We give examples of families of such solutions.

2. THE AXIOMS

We shall assume that the pair (a, S) satisfies the following, usual conditions:

ASSUMPTION 1: *There is at least one point $x \in S$ such that $x^i > a_i$ for $i = 1, 2$. In other words, bargaining may prove worthwhile to both players.*

ASSUMPTION 2: *S is convex. This is justified under the assumption that if two outcomes of the game give rise to points x and y in S, then randomizations of these two outcomes give rise to all convex combinations of x and y.*

ASSUMPTION 3: *S is compact.*

ASSUMPTION 4: *$a \leqslant x$ for every $x \in S$. If this is not the case, we can disregard all the points of S that fail to satisfy this condition because it is impossible that both players will agree to such a solution.*

We let U denote the set of pairs (a, S) satisfying these four conditions, and we call an element in U a *bargaining pair.*

A solution to the bargaining problem is a function $f: U \to R^2$ such that $f(a, S) \in S$. We shall confine ourselves to functions satisfying the following three axioms and we will call these functions *solutions.*

AXIOM 1—Pareto Optimality: *For every $(a, S) \in U$ there is no $y \in S$ such that $y \geqslant f(a, S)$ and $y \neq f(a, S)$.*

AXIOM 2—Symmetry: *We let $T: R^2 \to R^2$ be defined by $T((x_1, x_2)) = (x_2, x_1)$ and we require that for every $(a, S) \in U$, $f(T(a), T(S)) = T(f(a, S))$.*

AXIOM 3—Invariance with Respect to Affine Transformations of Utility: *A is an affine transformation of utility if $A = (A_1, A_2): R^2 \to R^2$, $A((x_1, x_2)) = (A_1(x_1), A_2(x_2))$, and the maps $A_i(x)$ are of the form $c_i x + d_i$ for some positive constant c_i and some constant d_i. We require that for such a transformation A, $f(A(a), A(S)) = A(f(a, S))$.*

In addition to the above three axioms, Nash introduced the following:

AXIOM OF INDEPENDENCE OF IRRELEVANT ALTERNATIVES: *If (a, S) and (a, T) are bargaining pairs such that $S \subset T$ and $f(a, T) \in S$, then $f(a, T) = f(a, S)$.*

He proved the surprising result that there is one and only one solution η which satisfies the axiom of independence of irrelevant alternatives. Nash's unique solution has the following very simple geometric interpretation: Given a bargaining pair (a, S), for every point $x = (x_1, x_2) \in S$, consider the product (area of a rectangle) $(x_1 - a_1) \cdot (x_2 - a_2)$. Then $\eta(a, S)$ is the unique point in S that maximizes this product.

Many objections were raised to Nash's axiom of independence of irrelevant alternatives (see, for example [**2**]). We shall raise another objection after introducing some additional notation. For a pair $(a, S) \in U$, let $b(S) = (b_1(S), b_2(S))$ be defined in the following way:

$$b_1(S) = \sup \{x \in R : \text{for some } y \in R \ (x, y) \in S\},$$

$$b_2(S) = \sup \{y \in R : \text{for some } x \in R \ (x, y) \in S\}.$$

Let $g_S(x)$ be a function defined for $x \leqslant b_1(S)$ in the following way:

$$\begin{aligned} g_S(x) &= y \quad \text{if } (x, y) \text{ is the Pareto of } (a, S), \\ &= b_2(S) \quad \text{if there is no such } y. \end{aligned}$$

Then $g_S(x)$ is the maximum player 2 can get if player 1 gets at least x. By Assumption 1 in the definition of a bargaining pair $b_i(S) > a_i$ for $i = 1, 2$. Also by the compactness of S, $b_1(S)$ and $b_2(S)$ are finite and are attained by points in S. A pair (a, S) will be called *normalized* if $a = 0 = (0, 0)$ and $b(S) = (1, 1)$. Clearly every game can be normalized by a unique affine transformation of the utilities. We can restrict our attention to the values that a solution takes on the normalized pairs, and Axiom 3 gives us a unique way to find the value of the solution for any non-normalized pair.

Consider the two normalized pairs $(0, S_1)$ and $(0, S_2)$ where

$$S_1 = \text{convex hull } \{(0, 1), (1, 0), (3/4, 3/4)\} \quad \text{and}$$

$$S_2 = \text{convex hull } \{(0, 1), (1, 0), (1, 0.7)\}.$$

For any fixed value of $x (0 < x < 1)$ there is a value of y for which (i) $(x, y) \in S_2$, and (ii) if $z \in R$ such that $(x, z) \in S_1$, then $y > z$. That is, $g_{S_1}(x) \leqslant g_{S_2}(x)$.

Based on these facts, player 2 has a good reason to demand that he get more in the bargaining pair $(0, S_2)$ than he does in $(0, S_1)$. Nash's solution of $(0, S_1)$ is $(\frac{3}{4}, \frac{3}{4})$ and his solution of $(0, S_2)$ is $(1, 0.7)$. These solutions do not satisfy player 2's demand.

In order to overcome this difficulty we suggest an alternative axiom.

AXIOM OF MONOTONICITY: *If (a, S_2) and (a, S_1) are bargaining pairs such that $b_1(S_1) = b_1(S_2)$ and $g_{S_1} \leqslant g_{S_2}$, then $f_2(a, S_1) \leqslant f_2(a, S_2)$ (where $f(a, S) = (f_1(a, S), f_2(a, S))$).*

This axiom states that if, for every utility level that player 1 may demand, the maximum feasible utility level that player 2 can simultaneously reach is increased, then the utility level assigned to player 2 according to the solution should also be increased.

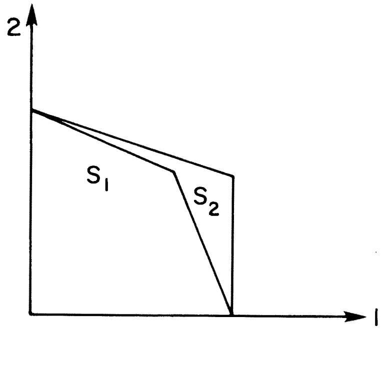

FIGURE 1

3. THE UNIQUE MONOTONIC SOLUTION

THEOREM: *There is one and only one solution, μ, satisfying the axiom of monotonicity. The function μ has the following simple representation. For a pair $(a, S) \in U$ consider the line joining a to $b(S)$, $L(a, b(S))$. The maximal element (with the partial order of R^2) of S on this line is $\mu(a, S)$.*

This solution was discussed in 1953 by Raiffa as a possible solution for the case when interpersonal comparison of utilities is allowed, and was arrived at experimentally by Crott [**1**].

PROOF OF THE THEOREM: We first show that μ is well defined. We let (a, S) be a fixed bargaining pair. $L(a, b(S))$ has a positive slope so that the partial order of R^2 induces a total order on $L(a, b(S))$. This implies that if $L(a, b(S))$ intersects S, then there is a unique maximal element of S on it, and $\mu(S)$ is well defined. The fact that $L(a, b(S))$ intersects S follows from the facts that there is a point $(b_1(S), y) \in S$ such that $y \geqslant a_2$; there is a point $(x_1, b_2(S)) \in S$ such that $x \geqslant a_1$, $a < b(S)$; and S is convex.

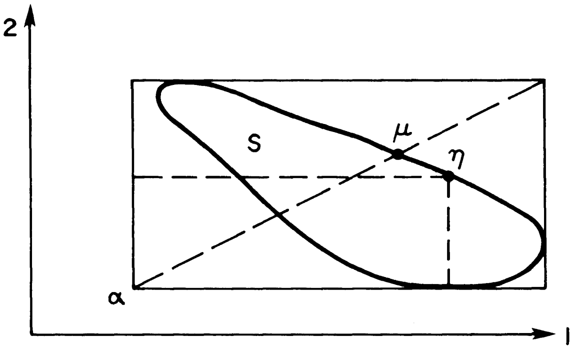

FIGURE 2

Next we have to show that μ is a solution and that it satisfies the axiom of monotonicity. It is easy to see that μ is symmetric. The fact that μ satisfies the axiom of Pareto optimality follows from the compactness and convexity of S. To see that μ is invariant under affine transformations of the utilities, we assume that A is such a transformation and that (a, S) is a bargaining pair. The following facts are true: (i) A preserves the partial ordering of R^2; (ii) A maps straight lines into straight lines; and (iii) A maps $b(S)$ into $b(A(S))$. These facts and the definition of μ imply that μ is invariant under affine transformations of the utilities.

The monotonicity follows from the following geometric observations. If L_α is a line of slope $\alpha (0 \leqslant \alpha \leqslant \pi/2)$ passing through a and if $(\sigma_1(\alpha), \sigma_2(\alpha))$ is the intersection point of the L_α with the boundary of $\{x \in R^2 : x \geqslant 0 \text{ and } x \leqslant y \text{ for some } y \in S\}$, then if $\beta > \alpha$, $\sigma_2(\beta) \geqslant \sigma_2(\alpha)$ and if $(\sigma_1^{(2)}(\alpha), \sigma_2^{(2)}(\alpha))$ is the corresponding point for (a, S_2), then $\sigma_1^{(2)}(\alpha) \geqslant \sigma_1^{(1)}(\alpha)$.

Finally, we prove that μ is the only solution that satisfies the condition of monotonicity. It is enough to prove this fact for normalized bargaining pairs. So let $(0, S)$ be such a pair and f any monotonic solution. Let $S_1 = \{x \in R^2 : x \geqslant 0$ and $x \leqslant y$ for some $y \in S\}$. Clearly $(0, S_1)$ is a normalized bargaining pair, $S_1 \supset S$, and there is no point $y \in S_1$ such that $y \geqslant f(0, S_1)$ and $y \neq f(0, S_1)$. Therefore $f(0, S_1) = f(0, S)$. Also the points $(0, 1)$ and $(1, 0)$ are in S_1. Let $S_2 =$ convex hull $\{(0, 1), (1, 0), \mu(0, S_1)\}$. Then $(0, S_2)$ is a normalized bargaining pair, it is symmetric for the two players, and $S_2 \subset S_1$. Therefore $f(0, S_2) = \mu(0, S_1)$. Also S_1 contains no point y such that $y \neq f(0, S_2)$ and $y \geqslant f(0, S_2)$. Therefore $f(0, S) = \mu(0, S_1) = \mu(0, S)$, and this completes the proof.

4. FURTHER DISCUSSION

There is an interesting duality relation between Nash's solution and the solution presented here. Let (a, S) be a bargaining pair. Let $\bar{S} = \{x \in R^2 : x \geqslant a$ and for some $y \in S, x \leqslant y\}$. Consider rectangles with sides parallel to the axes that are contained in $\bar{S}$. Nash's solution, η, is the maximal element on the southwest-northeast diagonal of the maximal area rectangle among all these rectangles. Now consider rectangles with sides parallel to the axes and which contain $\bar{S}$. The solution presented here, μ, is the maximal element on the southwest-northeast diagonal of the minimal area rectangle among all such rectangles.

Both Nash's solution and the solution presented here are continuous functions of the pairs (a, S). Since the condition that Nash imposed on his solution and the condition of monotonicity that we presented here may not be accepted by some people, a natural question arises: What are all the continuous solutions (in the sense defined here)? We know that η and μ presented here are not the only continuous solutions. New solutions can be obtained by taking various types of combinations of old solutions. Also given a solution σ one can obtain a whole family of new continuous solutions $F(\sigma)$ as follows. Because of the invariance under affine transformations of utilities it is enough to define a solution on all bargaining pairs normalized in a certain way. Given a solution σ, normalize every bargaining pair (a, S) to the bargaining pair $(0, S')$ in the unique way that carries $\sigma(a, S)$ to $(1, 1)$. Let G be a symmetric (in the two coordinates) probability measure on D, the quarter of the unit circle that lies in $R^2_+ = \{\theta \in R^2 : \theta > 0\}$. For every $\theta \in D$ let $x(\theta) = 0$ and $y(\theta) = 0$ if the line containing 0 and θ, $L(0, \theta)$, does not intersect S'. If the line $L(0, \theta)$ does intersect S', let

$$x(\theta) = \sup\{x \in R : \text{for some } y \in R, (x, y) \in S \cap L(a, \theta)\}, \quad \text{and}$$

$$y(\theta) = \sup\{y \in R : \text{for some } x \in R, (x, y) \in S \cap L(a, \theta)\}.$$

Let $\hat{S} = (\bar{x}, \bar{y})$ be defined by

$$\bar{x} = \int_{\theta \in D} x(\theta)\, dG(\theta),$$

and

$$\bar{y} = \int_{\theta \in D} y(\theta)\, dG(\theta).$$

Define $f_G(0, S')$ to be the maximal element of S' on the line $L(0, \hat{S})$. Then f_G defined this way turns out to be a continuous solution of the bargaining problem, and we define $F(\sigma) = \{f_G : G$ is a symmetric probability distribution on $D\}$. If G puts all its mass on $(\sqrt{\frac{1}{2}}, \sqrt{\frac{1}{2}})$, then $f_G = \sigma$. It is true that $\eta \notin F(\mu)$ and $\mu \notin F(\eta)$ so that neither $F(\eta)$ nor $F(\mu)$ contains all the continuous solutions.

An interesting problem is to try to classify all the possible continuous solutions. Solving this problem may lead us to alternative solutions to the bargaining problem as well as to better understanding of them.

Tel-Aviv University.

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