# Another View of Nonstandard Analysis

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... there are good reasons to believe that nonstandard analysis, in some version or other, will be the analysis of the future. *Kurt Gödel* [1973]

## 0. Introduction

In 1961, Abraham Robinson based a new way to study limits, continuity, and other aspects of analysis on Thoralf Skolem's nonstandard models for Peano arithmetic [1934], adding a sound generalization of Leibniz's infinitesimals to the 19th century epsilon-delta methods now considered standard. Generalizations of Robinson's nonstandard analysis have since been used in such diverse fields as functional analysis, number theory, probability, dynamical systems, and mathematical economics [1988]. Nonstandard analysis provides powerful new tools not only for proving or refuting conjectures and simplifying standard proofs, but also for giving precise meaning to many informal notions - like large integers and neighboring points - useful for constructing mathematical models for diverse phenomena and in teaching calculus, analysis, and topology. To make the subject more accessible to students and non-specialists, this note assumes no familiarity with the concepts from mathematical logic [1961], superstructures on ultrapowers [1962], or axiomatic set theory [1977] used in other approaches.

All versions of nonstandard analysis relate standard numbers to others in much the way that numbers like 1/7 and used in exact and symbolic computations relate to numbers like .142857 and 3.14159 used in numerical approximations. While nonstandard integers are too large to be uniquely specified, each has a decimal representatiion with a nonstandard number of digits, and students can compute with these in much the way that they do with standard integers, without reference to any formal theory; *e.g.* $(312...231)^2 = 97...361$. Each nonstandard positive integer exceeds all standard ones, and each has the familiar arithmetic properties of all standard integers; *e.g.*, each is a product of primes and a sum of four squares. Some mathematicians use Edward Nelson's Internal Set Theory [1977] to classify both standard and nonstandard integers as finite, and hence members of the ordered ring **Z** of finite integers. Others use a more traditional set theory to classify nonstandard integers as neither finite nor members of **Z**.

To combine certain desirable features of both classifications, we introduce the notion that nonstandard integers are guests of **Z** rather than nonstandard members, and that these integers, their reciprocals, and certain other nonstandard numbers are guests of the ordered field **R** of reals. We identify the guests of **R** with certain of the numbers introduced by J. H. Conway [1976], and view all these numbers as points on a continuous number line.

## 1. Surreals

We identify the numbers 0, 1, 2 ... with finite ordinals, non-negative integers, and natural numbers. Children learn to compute with these and other real numbers long before seeing axioms for ordered fields, or constructions using Dedekind cuts or Cauchy sequences. We can similarly learn to compute with other numbers before seeing axioms or constructions for them. In particular, the familiar addition and multiplication operations on finite ordinals extend naturally to infinite ones so that each set of ordinals generates a field. Differences of ordinals generalize integers, and ratios of these differences generalize rationals; *e.g.*, subtracting 1 from the least infinite ordinal yields the generalized integer - 1 which is not an ordinal, and dividing 1 by yields the generalized rational 1/ which is positive and less than all positive reals. Instead of using different constructions for integers, rationals, and other reals, J. H. Conway [1976] found a single construction for all these numbers and others, which Donald Knuth [1974] named "surreals". Harry Gonshor [1986] identified surreals with two-valued sequences, finite and infinite. The only properties of surreals used here are:

- All reals and all ordinals are surreal.
- Natural addition and multiplication of reals and ordinals extend to all surreals; and equipped with these operations, each set of surreals generates an ordered field of surreals.
- Each ordered field extending **R** is isomorphic to a surreal extension of **R**.

One example of an ordered field extending **R** consists of rational functions over **R** under pointwise addition and multiplication, ordered by f < g iff for some a **R**, f(x) < g(x) for all x > a. This extension is isomorphic to the ordered field **R**( ) of surreals generated by .

### Definitions

A surreal x is

- *-small* iff -1 < nx < 1 for all ordinals n < ,
- *-large* iff x has an -small reciprocal, 1/x, and
- *-near* surreal y iff x - y is -small.

No real is -large, the ordinal 0 is the only -small real, an ordinal is -large iff it is infinite, and a surreal is -large iff it is not between two reals. No real is -near another, each surreal is -near at most one real, and a surreal is -large iff it is -near no real. A surreal is -near a real iff no other real is between them, and a surreal is -near a non-zero real iff their ratio is -near 1 **R**.

# 2. *Reals

We assume that each infinite sequence of real numbers $s_n$ has a surreal **tag* $s_*$ satisfying these

### Axioms:

- If $r_n + s_n = t_n$ for all finite ordinals n then $r_* + s_* = t_*$.
- If $r_n\ s_n = t_n$ for all finite ordinals n then $r_*\ s_* = t_*$.
- If $s_n$ = x **R** for all finite ordinals n then $s_* = x$.
- If $s_n = n$ for all finite ordinals n then $s_*$ is -large.

Some consequences of these axioms are:

- Sequences with different *tags have infinitely many different terms.
- Sequences converging to x **R have *tags -near x.**
- **Sequences with *tags -near x R cluster about x.**

- **Sequences of zeros and ones have *tags which are zero or one.**
- **Sequences of zeros and ones with *tag 1 are the characteristic sequences for the members of a nontrivial ultrafilter on the set of finite ordinals.**
- **The map pairing each infinite sequence of reals with its surreal *tag is a ring homomorphism from the ring of infinite sequences onto a surreal field extending R.**

## Definitions

A surreal number is

- a **member* of X R iff it equals $s_*$ for some sequence s in X;
- a *guest* of X R iff it is a *member of X and not a member, and
- a **real* iff it is a *member of R.

All reals are *reals, all members of X R are *members, and a *member of X is a member iff it is real. A subset in R has guests iff it has infinitely many members, and subsets have different guests iff they have infinitely many different members. Each guest of R is a guest of at most one member of each partition of R, and a partition of R is finite iff each guest of R is a guest of some member of the partition.
Subset X R is

- discrete iff no guest of X is -near a member of X,
- a n'hood of x X iff no guest of the complement X' to X in R is -near x,
- open iff no guest of X' is -near a member of X,
- closed iff no guest of X is -near a member of X',
- compact iff each guest of X is -near a member of X, and
- perfect iff some guests of X are -near each member of X and none are -near members of X'.

# 3. Continuity and Differentiability

## Definitions

- The *composition* of sequence s in set X with map f:X Y is the sequence fs in set Y with $(fs)_n$ = $f(s_n)$ for all finite ordinals n.
- function f:X Y has **value* y at *member x of X iff for all sequences s in X, if x = $s_*$ then y = $(fs)_*$.

Each function between subsets in R has just one *value at each *member of its domain. No ambiguity results from using "f(x)" to denote both the value and *value of f:X Y at x, since these are equal whenever both are defined. All *values of f:X Y are *members of Y, and f(x) is a guest of Y iff x is a guest of X and of no level set of f in X. Functions with domain X R have the same *value at guest x of X iff they have the same restriction to some subset in X with guest x.
Subset X in R is a n'hood of x X iff x + x is a *member of X for all -small guests x of R. Function f:X R over a n'hood X of x is

- bounded on some n'hood of x iff f ( = f(x + x) - f(x) ) is -large for no -small guest x of R,

- continuous at x iff f is -small for all -small guests x of R,
- Lipshitz continuous at x iff f /x is -large for no -small guest x of R,
- differentiable at x with slope f'(x) R iff f /x is -near f'(x) for all -small guests x of R, and
- constant on some n'hood of x iff f = 0 for all -small guests x of R.

Function f:X R over an open subset X in R is continuous but not uniformly continuous iff for all -small guests x, f(x,x) is -small for all members x of X and not for some guests x of X. For example, the squaring function from R to R is continuous but not uniformly continuous since for all -small guests x, $(x^2)$ = 2x x + $(x)^2$ is -small for all x R but not for some -large guests x of R.

# 4. Transfer Principles

Theorems about reals and finite sets of reals generalize to theorems about *reals and finite sets of *reals; *e.g.*, a *real is non-negative iff it is the square of a *real, and each polynomial of odd degree with *real coefficients has a *real root. To generalize theorems about infinite sets, we assign a *tag $X_*$ to certain infinite sequences of nonempty sets $X_n$.

## Axiom

- If each sequence in set X has a *tag, then each sequence of nonempty subsets $X_n$ in X has the *tag $X_*$ = {$s_*$: $s_n X_n$ for all finite ordinals n}.

## Definitions

- Set X yields a *standard* $^*X$, iff each sequence in X has a *tag, and $^*X$ is the set of the *tags of all sequences in X.
- A subset in a standard $^*X$ is *internal* iff it is the *tag of some sequence of subsets $X_n$ in X..

The ordered field R of real numbers yields the standard ordered field $^*R$, consisting of the members and guests of R. If set X yields a standard $^*X$, then so does the set PX of all subsets in X, with $^*(PX)$ P($^*X$). Ordered by inclusion, standard subsets in $^*X$ form a Boolean subalgebra in the Boolean algebra $^*(PX)$ of internal subsets, which is in turn a Boolean subalgebra in the Boolean algebra P($^*X$) of all subsets in $^*X$; $^*(PX)$ = P($^*X$) iff X is finite.

If sets X and Y yield standard subsets $^*X$ and $^*Y$ in a standard set, then so do XY and XY; with $^*(XY)$ = $^*X^*Y$ and $^*(XY)$ = $^*X^*Y$.

Theorems about reals and sets of reals generalize to theorems about *reals and internal sets of *reals, provided each specified set of reals is replaced by its *std; *e.g.*, since each nonempty subset in the set N of finite ordinals has a least member, so each nonempty internal subset in the set $^*N$ has a least member, even though some non-internal subsets in $^*N$, like the set of all guests of N, have no least member.

# 5. References

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