Physics 213 2011 Assignment #10

Due Friday Dec. 19, 4 pm (turn in to envelope outside my office)

Reading: pp. 343-4, first two paragraphs of 10.2, 10.5, 10.8, start of 10.9 through p. 373

Individual problems (you may not consult with other students on these, but you may consult with me): None

Group problems (You are encouraged to work with other students on these _after_ you've put in a serious effort -- at least 10 minutes per problem -- on your own. You should avoid working in groups larger than 5 students.)

9.14 (2 points)
9.19 (2 points)
9.21 (a: 1 point, b: 3 points, c: 2 points)
10.7 (2 points)
10.9 (3 points)

10A. Use Mathematica for all parts of this problem. Start your Mathematica notebook with the following code:

```
$Assumptions = {t ∈ Reals,kc > 0, σ > 0}
m=9.1093826×10^-31   (* mass of electron, kg *)
ℏ=1.0545716×10^-34   (* Planck's constant J s *)
```

(To get the $\hbar$ symbol, type Esc-hb-Esc. To get the σ symbol, type Esc-s-Esc. The "×" symbol will appear automatically when you type the mantissa followed by a space followed by 10.)

a. (1 point) The quantum state of a particular electron at $t = 0$ is a Gaussian centered on $x = 0$, i.e. $\psi(x) = \psi_0\, \mathrm{e}^{-x^2/(2\sigma^2)}$. As you'll recall, the "probability density" is $|\psi(x)|^2$; the probability that the electron will be found between x and $x + \mathrm{d}x$ is $|\psi(x)|^2\,\mathrm{d}x$. Since the electron must be somewhere, the "normalization condition" is that $\int_{-\infty}^{\infty} |\psi(x)|^2\,\mathrm{d}x = 1$. Use this to show that $\psi_0 = \frac{1}{\pi^{1/4}\sqrt{\sigma}}$.

b. (1 point) Show that the Fourier transform of $\psi(x)$ is $Y(k) = \frac{\sqrt{\sigma}}{\pi^{1/4}}\mathrm{e}^{-k^2\sigma^2/2}$. As we've seen before, this means that there is an inverse relationship between the FWHM of $\psi(x)$ and the FWHM of $Y(k)$.

c. (1 point) Of course, from the definition of a Fourier transform pair, we could write $\psi(x)$ as $\psi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \mathrm{e}^{ikx}\,Y(k)\,\mathrm{d}k$. The sinusoids being used to synthesize $\psi(x)$ have a range of wavenumbers centered on 0. If we want instead to use a range centered on k_c, we would simply write $\psi(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx}\,Y(k-k_c)\,dk$, where $Y(k-k_c)=\frac{\sqrt{\sigma}}{\pi^{1/4}}e^{-(k-k_c)^2\sigma^2/2}$. (Note that this is a different $\psi(x)$ than what we started with in part a. In particular, the normalization is now slightly screwed up, but ignore this complication and continue to use the value for ψ_0 that you computed in part a.) To convert this to a traveling wave, we would replace *kx* by *kx* -ωt, giving $\Psi(x,t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i(kx-\omega t)}\,Y(k-k_c)\,dk$. First, define $\omega(k)$ as is appropriate for an electron, then type the expression for $\Psi(x,t)$ into Mathematica without delayed evaluation (i.e. use Ψ[x_, t_] = …, rather than Ψ[x_, t_] := …), and evaluate it. You should get something rather messy in terms of the symbols σ and k_c.

d. (1 point) If we want the electron to travel at speed v_e, what value should we use for k_c?

e. (1 point) Set v_e = 50,000 m/s and the initial FWHM of $\Psi(x,t)$ to be 100 nm. (Recall that 1 nm = 10^{-9} m, and that the FWHM of a Gaussian equals 2.35 σ.) Plot the magnitude of $\Psi(x,0)$ from x = -300 nm to 600 nm, and use the GridLines command to draw vertical lines at the expected values of x where the magnitude should fall to half the peak value. The syntax would be:

```
Plot[Abs[Ψ[x, 0]], {x, −3 10^−7, 6 10^−7}, PlotRange → All, GridLines → {{xvalue, xvalue}, {}}]
```

where you are to fill in the two "xvalue" entries. Note that an entry such as "3 10^{-7}" means "3×10^{-7}" -- the space (which doesn't show up well in my copy-and-paste version shown above) indicates multiplication, so the plot range is -3×10^{-7} to 6×10^{-7}. Because you are plotting the magnitude of $\Psi(x,t)$, you are plotting only the envelope of the wavepacket, not the short-wavelength ripples inside the envelope. Verify that the plot looks the way you expect.

f. (1 point) Now make an animated version of the plot, in which the time runs from 0 to 10 ps. (Recall that 1 ps = 10^{-12} s.) You can omit the gridlines. The syntax would be

```
Animate[Plot[Abs[Ψ[x, t]], {x, −3 10^−7, 6 10^−7}, PlotRange
        → {0, entermaxvalue}], {t, 0, 10^−11}]
```

For "entermaxvalue" you enter the maximum vertical value of the plot from part e, to keep the plot from constantly changing scales as it does the animation. Use the button with the double down-arrow to slow the animation down. Roughly measure the speed of the peak by eye, report your finding, and compare it with your expectation. Also comment qualitatively on whether the width of the peak changes over the time interval you've animated, and if so give a qualitative indication of by how much (e.g. "a huge amount", "a tiny smidge", etc.).

g. (1 point) Repeat steps e and f using a FWHM of 10 nm, and omitting the GridLines. (They will be too close to the origin to be useful for such a small FWHM.) You should not need to go back and redefine $\Psi(x,t)$ -- just enter the new value of σ and re-do the plot and the animation. Note that you'll need to change the vertical scale for the animation.

h. (2 pionts) You should have seen a dramatic change in the animations for the two different values of FWHM. Comment qualitatively on why the changes you see make sense, using ideas from our discussion in class of group velocity and the requirements on the dispersion relation for the envelope to propagate without changing its shape.

i. (2 points) Change the FWHM back to 100 nm, and re-do the animation from part f, but this time plot the real part of $\Psi(x,t)$, with a zoom in on both the space and time ranges:

```
Animate[Plot[{Re[Ψ[x, t]]}, {x, −10^−7, 3 10^−7}, PlotRange
        → {−entermaxvalue, entermaxvalue}], {t, 0, 3 10^−12}]
```

Again, you are to fill in the value for "entermaxvalue". Note that you need to change the lower limit for the plot range, because the real part can be negative. If you look carefully (after slowing down the animation), you should see the individual peaks within the packet move backward relative to the motion of the envelope, i.e. they are moving forward, but not as fast as the envelope. Comment on why this makes sense, based on our discussions in class.