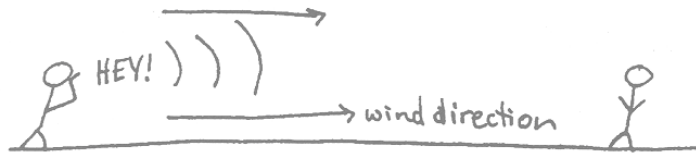


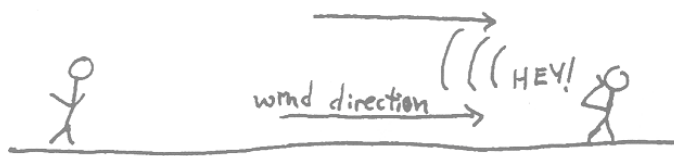
**Relativity fundamentals explained well (I hope)**  
**Walter F. Smith, Haverford College 3-14-06**

**Propagation of waves through a medium**

As you'll recall from last semester, when the speed of sound is measured relative to the ground, it is faster for sound waves traveling downwind than upwind:



Sound travels quickly downwind

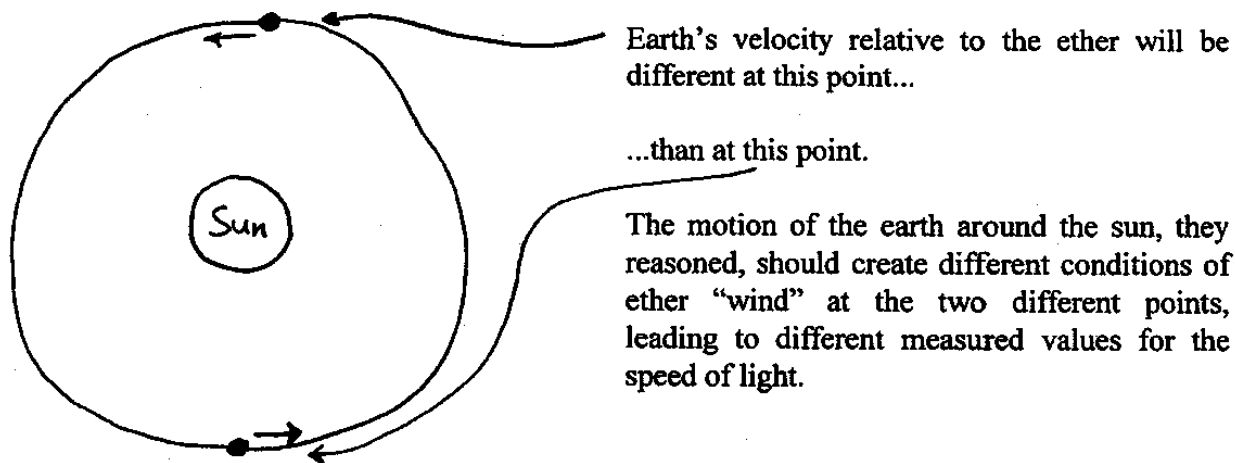


Sound travels more slowly upwind

The air is the “medium” which carries the sound.

**The Michaelson-Morley experiment**

In the late 1800's, it was universally believed that light waves traveled through a medium as well; the medium was called the “ether.” It was assumed that the ether was at rest with respect to the center of the universe, or perhaps the center of the galaxy. In 1887, A. A. Michaelson and E. W. Morley set out to measure differences in the speed of light caused by the motion of the earth relative to the ether:



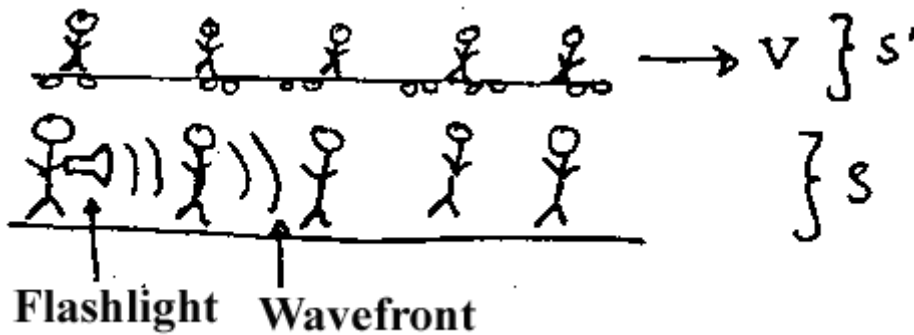
They made extremely precise measurements, and always found exactly the same value for the speed of light. The only logical conclusion was that light does not need a medium to travel through! It can travel through vacuum! Another way of saying this is that the speed of light measured by any experimenter will always be the same, whether the experimenter is moving to the right, to the left, or is still. In fact, as we've seen, Maxwell's equations show that the propagation of light is a basic form of electromagnetism, which propagates at a speed  $c = 1/\sqrt{\epsilon_0\mu_0}$ . Note that the speed of the observer doesn't appear in this equation, just as it doesn't appear in  $F = ma$ . So, in the same way that  $F = ma$  works in all constant velocity (or “inertial”) reference frames,  $c = 1/\sqrt{\epsilon_0\mu_0}$  works in all reference frames, i.e., light propagates with the same speed in all reference frames. (Again, this was shown by the Michaelson-Morley experiment.)

### The basic postulate of relativity:

*The laws of physics work equally well in all inertial reference frames. There is no preferred reference frame.*

This includes the propagation of light, since as we discussed above, light propagation is a consequence of Maxwell's equations, and since the Michelson-Morley experiment showed that light propagates at the same speed in a variety of reference frames.

This postulate has immediate counterintuitive consequences. For example, imagine two groups of observers. One group (S) is "stationary," while the other group (S') moves to the right at speed  $V = 0.9$  times the speed of light, i.e.,  $V = 0.9 c$ . One of the S observers turns on a flashlight, and the other S observers measure the speed at which the wavefront propagates:



The S observers, of course, measure a speed of  $c$  for the wavefront. What speed do the S' observers measure? In the way we're accustomed to think, they would measure a speed of  $1.0 c - 0.9 c = 0.1 c$ . However, this is wrong. The light is propagating in the S' frame as well as the S frame, so it must move with a speed of  $c$  in S' as well as in S!

In fact, as we'll see, this strange way of adding velocities is not unique to light propagation. Similar effects occur for any object moving very close to the speed of light.

For example, we will show that if the observers in S throw a rock to the right with speed (measured in S) of  $0.99 c$ , then the observers in S' will measure a speed of  $0.83 c$  for the rock, instead of the speed of  $0.09 c$  that one might expect. (You can see that the effect is most extreme for the case of something propagating at exactly  $c$ , since both sets of observers measure the same speed for it, despite their large relative velocities, while on the example we just did they measure similar but not identical velocities ( $0.99 c$  in S and  $0.83 c$  in S').

### The Four Basic Results

There are four basic results from special relativity. We'll derive them in the later sections of this document, but first let's just look at them, and see how they fit together to form a self-consistent picture:

1)  $V = \frac{\text{distance}}{\text{time}}$ , so long as all three are measured in the same reference frame. You already know this one, but it's reassuring to know that it still works, even when high speeds are involved.

2) Time Dilation:  $\Delta t_{\text{other}} = \gamma \Delta t_{\text{proper}}$ , where  $\Delta t_{\text{proper}}$  is the time interval between two events **as measured in the frame for which both events occur at the same place**,  $\Delta t_{\text{other}}$  is the time interval between these two events as measured in some other reference frame, and  $\gamma$  (the Greek letter "gamma") is defined as

$\gamma \equiv \frac{1}{\sqrt{1 - \frac{V^2}{c^2}}}$ . Note that when  $\gamma$  is written by hand, it looks like this:  $\gamma$ . Finally,  $V$  is the velocity of the

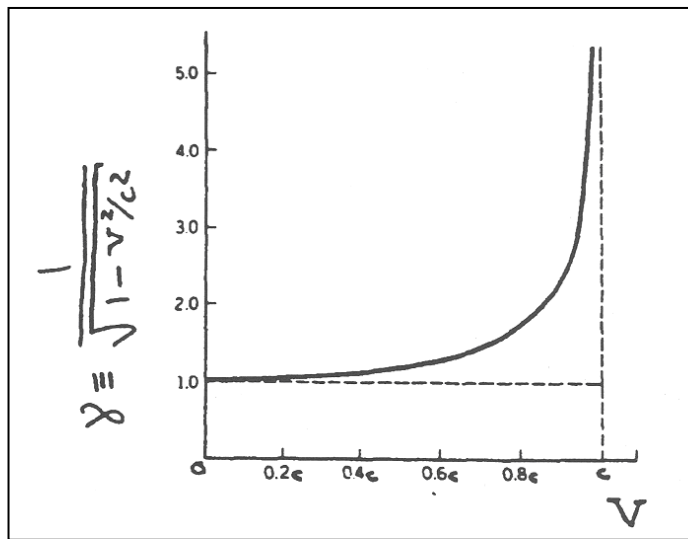
“other” frame relative to the “proper” frame. For example, say I’m moving to your right at constant speed, and I throw a piece of chalk into the air and then catch it. For me, the two events of 1) throwing the chalk and 2) catching it occur at the same place (at my hand), whereas for you they occur in different places. Therefore, I measure  $\Delta t_{\text{proper}}$  between these two events, and you measure  $\Delta t_{\text{other}}$ .

As we’ll see eventually,  $V$  is always less than or equal to  $c$ , so  $\gamma$  is always greater than 1. This effect only become easily noticeable when  $V$  is greater than about  $0.1c$ , as you can see from the plot of  $\gamma$  shown here.

### Verification Of Time Dilation

[This paragraph is taken from a textbook.] A striking confirmation of time dilation was achieved in 1971 by an experiment carried out by J.C. Hafele and R.E. Keating. They transported very precise cesium-beam atomic clocks around the world on commercial jets.

Since the speed of a jet plane is considerably less than  $c$ , the time-dilation effect is extremely small. However, the atomic clocks were accurate to about  $\pm 10^{-9}$  s, so that the effect could be measured. The clocks were in the air for 45 hours, and their times were compared to reference atomic clocks kept on earth. The experimental results revealed that, within experimental error, the readings on the clocks on board the planes were different from those on earth by an amount that agreed with the prediction of relativity.



An especially important example of time dilation is for two subsequent ticks of my watch (again, assume I’m moving to your right at constant speed). For me, the two ticks occur at the same place (on my wrist), so I measure  $\Delta t_{\text{proper}} = 1$  second between ticks, and you measure  $\Delta t_{\text{other}}$ . If  $\gamma = 5$ , then you would measure 5 seconds between my ticks. You would say that my watch is running slow by a factor of 5. In fact, everything about me is moving slowly, including my heartbeat, the rate of chemical reactions in my body, the rate of my aging, etc. So, the Time Dilation effect can also be stated as, **“The other person’s clock runs slow by a factor  $\gamma$ ”**

The really odd thing about this is that, of course, it works both ways – you see my watch running slow, but I see your watch running slow! This should seem paradoxical to you. We can only resolve the paradox by understanding effects 3 and 4 below, and then seeing how everything fits together.

3) Fitzgerald contraction:  $L_{\text{other}} = \frac{L_r}{\gamma}$ , where  $L_r$  is the “rest length” of an object, *i.e.* the length of the object in the frame for which the object is at rest, and  $L_{\text{other}}$  is the length measured in another frame. (The factor  $\gamma$  is as defined before, where now  $V$  is the velocity of the object.) In words, this says that the length of an object along the direction of travel shrinks by a factor  $\gamma$ .

4) Synchronization: The “chasing clock” leads by  $\Delta t_{\text{synch}} = \frac{L_r V}{c^2}$ . Imagine that frame  $S'$  is moving to your right at speed  $V$ . The people in  $S'$  have two clocks, with a separation as measured by them of  $L_r$ . They have carefully synchronized these clocks. However, this equation says that, to you the clocks are mis-synchronized, so that, at any instant, the one on the left (which appears to “chase” the one on the right) reads a time  $\Delta t_{\text{synch}} = \frac{L_r V}{c^2}$  greater than the one on the right. This is the effect which most people find the most confusing.

However, special relativity is only internally consistent when you include this effect. In particular, you can only resolve the time dilation paradox mentioned above by thinking about this synchronization effect.

### How the four effects fit together to resolve the time dilation paradox

Again, let's assume that people in frame S are "stationary", while those in S' move to the right at speed  $V$ . (Of course, from the equally-valid point of view of the folks in S', they're the ones that are stationary, and the people in S are moving to the left at speed  $-V$ .) How can it be that people in S see the clocks in S' running slow, but people in S' see the clocks in S running slow?

The best way to answer this is with a numerical example. Since things will be moving fast, we need to have large distances, so we'll measure distance in units of "light seconds". One light second is the distance that light travels in one second, and is written "1 cs". One of the nice things about expressing distances in these units is the way the factors of  $c$  cancel out. For example, to find the time it takes light to travel one light second:

$$\text{second: } \text{time} = \frac{\text{distance}}{\text{velocity}} = \frac{1 \text{ cs}}{c} = 1 \text{ s}, \text{ where in the last step we simply cancel the "c" in the numerator and}$$

denominator. In frame S', which travels to the right at speed  $V$ , let observer A' be 15 cs to the left of O' (as measured in S'). There is a single observer O in frame S. All three observers have digital clocks with readouts in seconds, as shown by the square boxes in the drawings on the next page.

There are two events of interest: 1) when O' passes O and 2) when A' passes O. When two observers pass by each other (e.g. when A' passes O), they both agree on what their clocks read, but their explanations for these readings are quite different as we'll see. We arrange things so that, when O' passes O both their clocks read 0. The two clocks in S' are synchronized (as seen by observers in S'). We'll pick  $V = \frac{3}{5}c$ , which gives

$$\gamma = \frac{1}{\sqrt{1 - \frac{V^2}{c^2}}} = \frac{5}{4}.$$

You can pretty well see from the drawing how things fit together, but if you like I'll walk you through it in the following paragraphs. Let's start with the upper left drawing, which shows the situation as seen in S' at event 1 (when O passes O'). We've arranged things so that the clocks of O and O' both read 0 for this event. Since the clocks in S' are synchronized, the clock of A' reads the same as that of O', *i.e.* it reads 0.

Moving down to the lower left drawing, we wait for O to move from O' to A'. O must cover a distance of 15 cs, and is moving at a speed of  $\frac{3}{5}c$ , so this takes 25 s. Thus, for event 2 (when O passes A'), the clocks of A' and O' both read 25 s. The clock of O started out (at event 1) reading 0, and it runs slow by a factor  $\gamma$ . Thus, during the 25 s that have elapsed, it only ticks off  $\frac{25 \text{ s}}{\gamma} = 20 \text{ s}$ , giving a final reading of  $0 + 20 \text{ s} = 20 \text{ s}$ .

The upper right drawing shows the situation as seen in S for event 1. The clocks of O and O' both read 0, as we've arranged. However, A' has the "chasing clock", so it leads the clock of O' by  $\Delta t_{\text{synch}} = \frac{L_r V}{c^2} = 9 \text{ s}$ .

The distance between A' and O' is Fitzgerald-contracted to  $L_{\text{other}} = \frac{L_r}{\gamma} = 12 \text{ cs}$ .

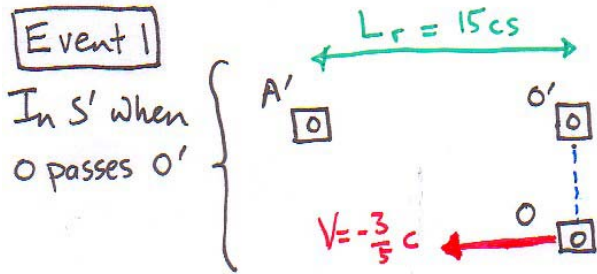
Moving down to the lower right drawing, we wait for A' to move to O. A' must cover a distance of 12 cs, and is moving at a speed of  $\frac{3}{5}c$ , so this takes 20 s. Thus, for event 2 (when A' passes O), the clock of O reads 20s. The clock of A' started out (at event 1) reading 9 s, and it runs slow by a factor  $\gamma$ . Thus, during the 20 s that have elapsed, it only ticks off an additional  $\frac{20 \text{ s}}{\gamma} = 16 \text{ s}$ , giving a final reading of

$(9 \text{ s}) + (16 \text{ s}) = 25 \text{ s}$ . (The clock of O' also advances by 16 s during the interval between events 1 and 2.)

Note that, for event 2, everyone agrees what is shown on the clocks of A' and O, but their explanations are quite different. The observers in S agree that the clock of A' shows a larger reading than that of O, but only

because it was mis-synchronized so that it started with a reading of 9 s instead of 0 s – they say that the clock of A' runs slow, since it showed only 16 s elapsing, when the actual time interval between the two events was 20 s. On the other hand, the people in S' say that the actual time interval was 25 s, and that the clock of O only shows a reading of 20 s because it runs slow.

As Observed in S'

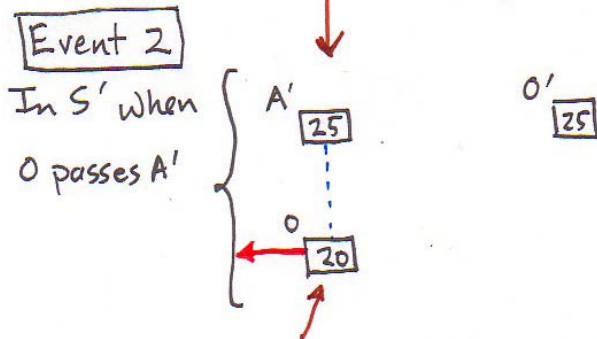


time passes  

$$\text{time} = \frac{\text{distance}}{\text{speed}}$$

$$= \frac{15cs}{\frac{3}{5}c}$$

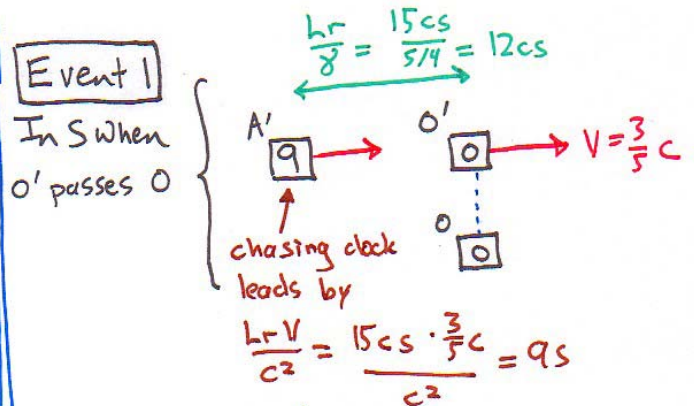
$$= 25s$$



$$= 0 + \frac{25s}{\gamma} = \frac{25s}{5/4} = 20s$$

initial reading      amount the clock of O advances in 25s. (It runs slow by a factor  $\gamma$ .)

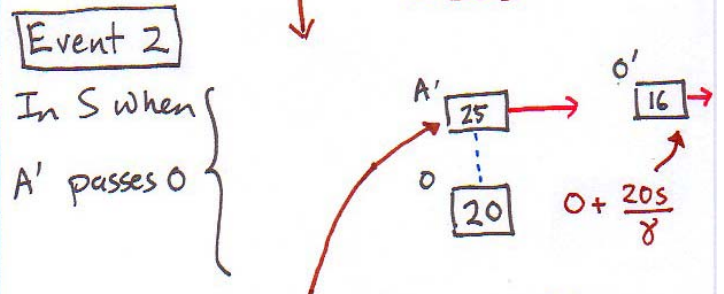
As Observed in S



time passes      time = distance / speed  

$$= \frac{12cs}{\frac{3}{5}c}$$

$$= 20s$$



$$= 9s + \frac{20s}{\gamma} = 9s + \frac{20s}{5/4} = 25s$$

initial reading      amount the clock of A' advances in 20s. (It runs slow by a factor  $\gamma$ .)

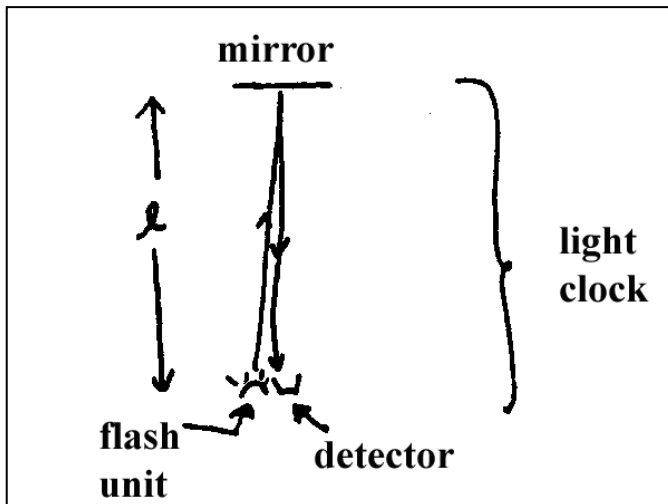
## Derivations of the four fundamental effects

1)  $V = \frac{\text{distance}}{\text{time}}$ . This is really just the definition of velocity.

### 2) Time Dilation

#### The light clock

Part of the reason that velocities close to  $c$  don't add in the way we expect is that time is perceived differently in  $S$  and  $S'$ . To investigate this, we'll use an unusual clock, the "light clock":



A device sends out a flash of light which travels upward, bounces off a small mirror, and then returns to a detector, which is at the same position as the flash unit. (For graphical clarity, the detector is shown just to the right of the flash unit.) As soon as this detector sees the reflected light flash, it triggers another flash. Each of these cycles is one "tick" of the light clock. Let's put one of these light clocks in the "moving" frame  $S'$ . We'll show that the rate at which this clock ticks, as perceived by the observers in  $S$ , depends on  $V$ , the relative velocity between  $S'$  and  $S$ .

#### Light clocks vs. ordinary clocks

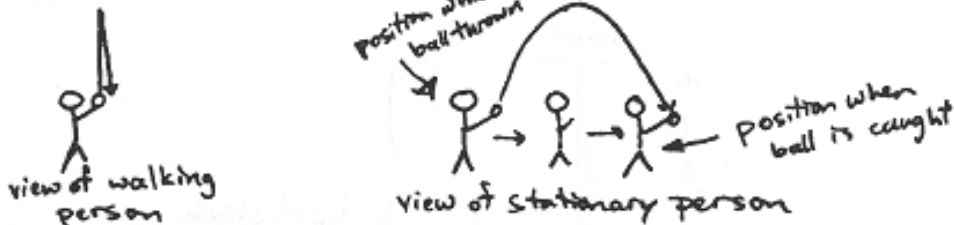
It's important to realize that the results we'll get are not limited to light clocks; any other clock in  $S'$  would display exactly the same variation. To see this, assume that the person in  $S'$  is initially at rest relative to  $S$ . She has a conventional clock, which she adjusts so that it has the same tick rate as the light clock. Now she starts moving. Since there is no preferred reference frame (by the basic postulate of relativity), there should be no way for her to tell that she was previously stationary and is now moving, rather than the other way around. For example, since the regular clock and the light clock were synchronized when she was "stationary," they should remain synchronized now that she is "moving." We could make a similar argument using her heartbeat. If there are a certain number of ticks of the lightclock per heartbeat when she is "stationary," there must be the same number when she is moving. It is still possible that the rate at which all these clocks tick (the light clock, the regular clock, and her heart) might vary in *unison*, as seen by the people in  $S$ . All that the person in  $S'$  can tell is that the clocks remain synchronized.

Since we could make these arguments using a chemical reaction or any other time-dependent phenomenon instead of her heartbeat, we see the results we will derive for the variation in the tick rate of the light clock in  $S'$  (as seen by the people in  $S$ ) are not limited to the behavior of the light clock itself, but are actually statements about the way time itself is passing in  $S'$  (as seen by the people in  $S$ ).

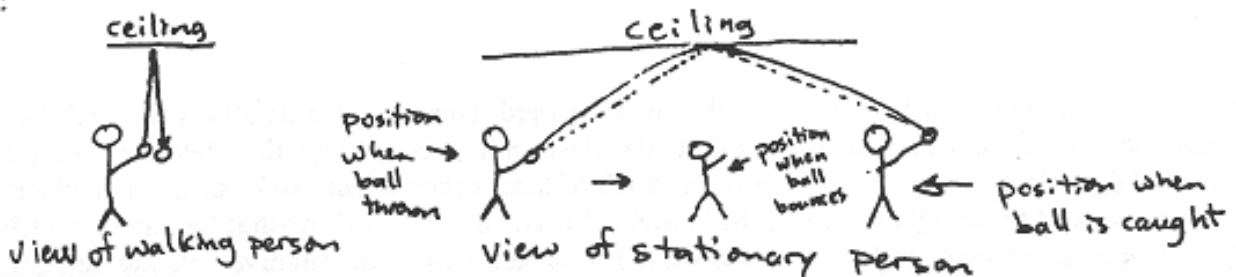
### Derivation of time dilation

First, let's think about the path followed by the light, as seen in  $S'$  and then as seen in  $S$ . The situation is very similar to a person walking at constant speed who throws a ball straight up (as seen by the walking person) into the air, and then catches it. To the person who is walking, the ball goes straight up and then straight down. However, to a stationary observer, the ball follows

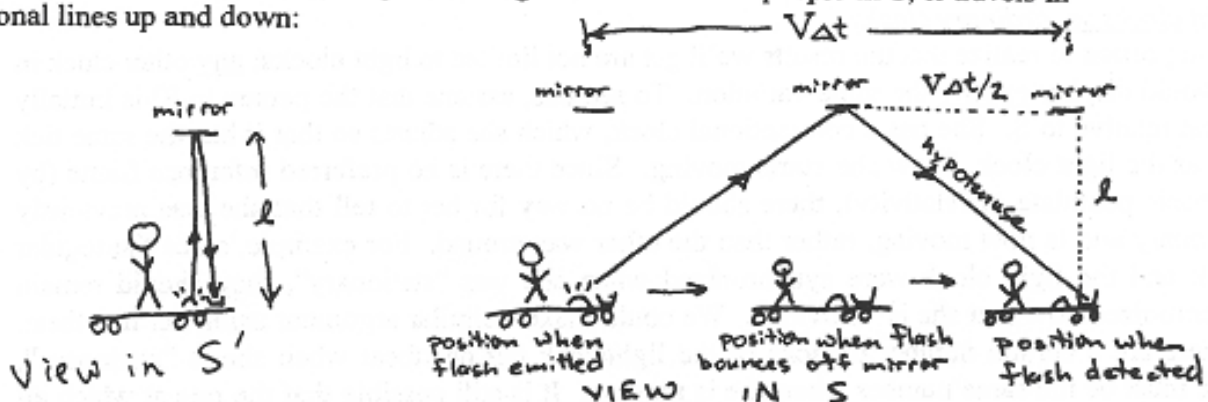
a parabolic trajectory:



Now imagine that the moving person is moving rather quickly, and throws the ball straight up, but quite fast. In fact, the ball bounces off the ceiling and then back down. Again, as seen by the moving person the ball goes straight up, bounces, then comes straight back down. However, for the stationary person, the ball, if thrown very fast, travels almost in straight diagonal lines up and down:



Finally, let's look at the path of the light flash in the light clock. This is like a very fast ball. To the person in  $S'$ , it goes straight up and straight down. For the people in  $S$ , it travels in diagonal lines up and down:



Let  $\Delta t$  be the time interval between when the flash is sent out and when it is received (i.e. the time interval between light clock ticks), as measured in  $S$ . Since the clock (and everything else in  $S'$ ) is moving at speed  $V$ , the light pulse must cover a horizontal distance of  $V\Delta t$ , as shown above. The total path length covered by the light is then the sum of the hypotenuses as shown above:

$$\text{path length (in } S) = 2\sqrt{l^2 + \left(\frac{V\Delta t}{2}\right)^2}$$

Since the light travels at speed  $c$ , the time that it takes to cover this path, which is equal to  $\Delta t$ , is given by

$$\Delta t = \frac{\text{path length (in S)}}{c} \Rightarrow (\Delta t)^2 = \frac{4}{c^2} \left[ \ell^2 + \left( \frac{V\Delta t}{2} \right)^2 \right] = \frac{4\ell^2}{c^2} + \frac{V^2}{c^2} (\Delta t)^2$$

However, it is also true that light travels with speed  $c$  in  $S'$ , so we can use a similar method to find the time  $\Delta t'$  between ticks as measured in  $S'$ :

$$\Delta t' = \frac{\text{path length (in S')}}{c} = \frac{2\ell}{c} \Rightarrow (\Delta t')^2 = \frac{4\ell^2}{c^2}$$

We can see right away that this is smaller than  $(\Delta t)^2$ , i.e. that the time between ticks as measured by the person in  $S'$  is shorter than the time between ticks as measured by the people in  $S$ ! Let's get more quantitative. Substituting our expression for  $(\Delta t')^2$  into the equation for  $(\Delta t)^2$  gives

$$(\Delta t)^2 = (\Delta t')^2 + \frac{V^2}{c^2} (\Delta t)^2 \Leftrightarrow (\Delta t)^2 \left( 1 - \frac{V^2}{c^2} \right) = (\Delta t')^2 \Rightarrow \Delta t = \frac{1}{\sqrt{1 - V^2/c^2}} \Delta t'$$

We'll encounter that square-root factor a lot, so we define

$$\gamma \equiv \frac{1}{\sqrt{1 - V^2/c^2}} \Rightarrow \Delta t = \gamma \Delta t'$$

As we'll see eventually,  $V$  is always less than or equal to  $c$ , so  $\gamma$  is always greater than 1. This equation says that the time between ticks as measured in  $S$  is *greater than the time between ticks as measured in  $S'$*  by a factor of  $\gamma$ ! The faster  $S'$  is moving, the greater the size of this effect. Let's assume that when everyone is at rest, their hearts beat at the same rate. Since the "dilation of time" derived above applies to all "clocks" in  $S'$ , including the heartbeat of the person in  $S'$ , this means that, as measured by the people in  $S$ , there is a longer time between the heartbeats of the person in  $S'$  than between their own heartbeats, and the faster she moves the longer this time becomes. Thus (according to the people in  $S$ ), the person in  $S'$  is aging more slowly than they are!

### Proper time

We just showed that  $\Delta t = \gamma \Delta t'$ . However, this seems to contradict the fundamental postulate of relativity, since the equation is not symmetrical between the two reference frames; the time interval as measured in  $S$  is longer than that measured in  $S'$ , and the faster  $S'$  goes, the more dramatic this effect becomes. However, there is something about the experiment with the light clock itself which makes a fundamental distinction between the two reference frames. The time interval  $\Delta t$  represents the time interval between two events: the first event is the flash, and the second is the reception of the flash. The fundamental distinction between the reference frames is that in  $S'$  these two events occur *at the same place* (because the flash unit and the detector are in the same place), while in  $S$  they occur at different places.

Thus, if we instead did the experiment with the light clock in  $S$  (instead of  $S'$ ) then the roles of the two reference frames would be reversed, and we would find  $\Delta t' = \gamma \Delta t$ , i.e. that the time as measured in  $S'$  between the two events is longer than the time as measured in  $S$ . So, the two reference frames really are equally good, it just depends on how we do the experiment.

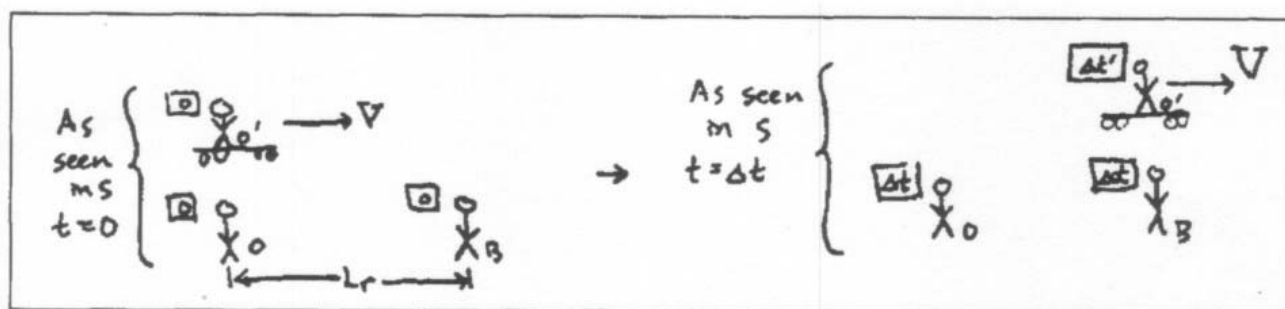
We note that the shorter time is always measured in the reference frame in which the two events occur at the same place: When the clock was in  $S'$ , the shorter time was measured in  $S'$ , whereas when the clock was in  $S$ , the shorter time was measured in  $S$ . **We define the "proper time" between two events to be the time as measured in the reference frame in which the two events occur at the same place.** With this definition, we can summarize all such experiments with a single equation:

$$\Delta t_{\text{other}} = \gamma \Delta t_{\text{proper}}$$

where  $\Delta t_{\text{proper}}$  is the proper time (from now on, we'll simply write it as  $\Delta t_p$ ), and  $\Delta t_{\text{other}}$  is the time as measured in some other reference frame. For example, if we do the experiment with the light clock in  $S$ , then both events (the flash and the reception) occur at the same place in  $S$ , so  $\Delta t_p = \Delta t$  and  $\Delta t_{\text{other}} = \Delta t'$ . However, if we do the experiment with the light clock in  $S'$  then both events (the flash and the reception) occur at the same place in  $S'$ , so  $\Delta t_p = \Delta t'$  and  $\Delta t_{\text{other}} = \Delta t$ .

### 3) FITZGERALD CONTRACTION

There is still an inconsistency in our theory. Everything should be symmetrical between frames  $S$  and  $S'$ . Therefore, the people in  $S$  and  $S'$  should agree on the magnitude of their relative velocity (even though they may disagree about who's moving). Let's consider a situation similar to our previous setup, but this time, there is only one observer ( $O'$ ) in frame  $S'$ , and there are two observers ( $O$  and  $B$ ) in frame  $S$ , separated by a distance  $L$  (as measured in frame  $S$ ):



The  $S$  observers can measure the velocity of  $S'$  by taking the distance travelled by  $O'$  and dividing it by the time interval, i.e.  $V = \frac{L_r}{\Delta t}$ . From the point of view of  $O'$ , the people in  $S$  are moving to the left, so she can use a similar method to calculate their velocity, i.e.  $V = \frac{L'}{\Delta t'}$ , where  $L'$  is the distance between  $O$  and  $B$  as measured by observers in  $S'$ . (From what we have derived so far, we would expect that  $L' = L_r$ , but we'll soon see that this is wrong, as was foreshadowed on p. 2.) Again, the magnitude of  $V$  as measured by the people in  $S'$  must be the same as that measured in  $S$ , i.e.

$$\frac{L'}{\Delta t'} = \frac{L_r}{\Delta t}$$

As we showed before, for the situation shown above,  $\Delta t = \gamma \Delta t'$ . Substituting this in, we get

$$\frac{L'}{\Delta t'} = \frac{L_r}{\gamma \Delta t'} \Leftrightarrow L' = \frac{L_r}{\gamma}$$

In other words, the  $S'$  observers perceive the distance between  $O$  and  $B$  to be less than the  $S$  observers do!! As we did for the case of time dilation, we can make a more generally useful equation by using  $L_{\text{other}}$ :

$$L_{\text{other}} = \frac{L_r}{\gamma}$$

#### 4) Synchronization

Let's return to the setup from page 4, but this time we'll keep all the variables in symbol form, rather than using particular numeric values. As before, we set things up so that the spacing between A' and O' is  $L_r$  and the clocks of A' and O' are synchronized, as seen in frame S'. Also, we arrange things so that the clocks of O and O' both read 0 at the moment that O' passes O. Finally, now A' has a green flashbulb.

As shown in the top right picture on the next page, as seen in S, the bulb goes off at the instant that O' passes O, i.e. at  $t = 0$ . Also, as seen in S, the spacing between A' and O' is Fitzgerald contracted to  $\frac{L_r}{\gamma}$ .

Therefore, the distance along the  $x$ -axis from the point of the flash to O is equal to  $\frac{L_r}{\gamma}$  -- we use this again in the bottom picture of the right column. In S, of course, the clocks of both O and O' read 0 at this instant, but observers in S see that the clock of A' reads a value greater than zero. We'll call this reading  $t'_f$  -- this is the time displayed on the clock of A' when she sets off her flashbulb, and this is our desired unknown. For a classical situation, we would have  $t'_f = 0$ , so that the clocks of A' and O' would appear synchronized even in S, but we'll show that  $t'_f > 0$ .

As shown in the top left picture, as seen in S' the clocks of A' and O' are synchronized. Thus, when O passes O', which happens when the clock of O' reads 0, the clock of A' also reads zero. Assuming for the moment that I'm right in asserting that  $t'_f > 0$ , this means that A' has not yet set off the flash, and that in S' the flash is set off after O passes O', as shown in the second picture down on the left.

Continuing down the left column of pictures, the flash spreads out at speed  $c$  from A', and eventually it passes O' at a time (as measured in S') that we'll call  $t'_c$ . Because the light had to travel a distance  $L_r$  to reach O', she can infer that A' must have set off the flash at

$$t'_f = t'_c - \frac{L_r}{c} \quad (1)$$

Now let's go back to the right column. After the flash is set off, the light must travel away from the flash point at speed  $c$ . Thus, it spreads out in circles that are centered on the point of origin of the flash, as shown in the middle picture on the right, rather than being centered on A', who is moving to the right.

As shown in the bottom picture on the right, eventually the flash catches up with O'. This happens at time  $t_c$ , where the "c" subscript stands for "catch up". During the time from  $t = 0$  to  $t = t_c$ , the light travels a distance  $ct_c$ , while O' travels a distance  $Vt_c$ . As shown in the figure,

$$ct_c = \frac{L_r}{\gamma} + Vt_c \Leftrightarrow t_c = \frac{L_r}{\gamma(c-V)} \quad (2)$$

Now, during the time interval between  $t = 0$  and  $t = t_c$ , the observers in S have seen that the clock of O' runs slow by a factor  $\gamma$ . Therefore, the time it now shows is

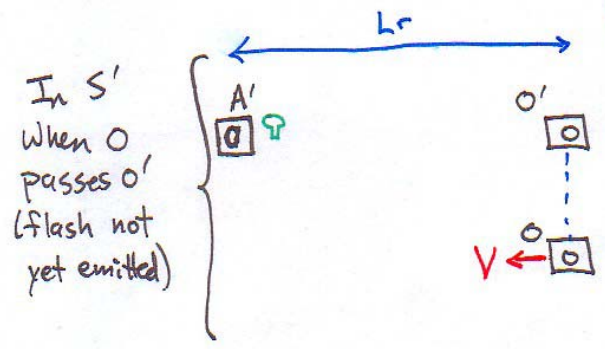
$$t'_c = \frac{t_c}{\gamma} = \frac{L_r}{\gamma^2(c-V)} \quad (3)$$

Substituting this into (1) gives

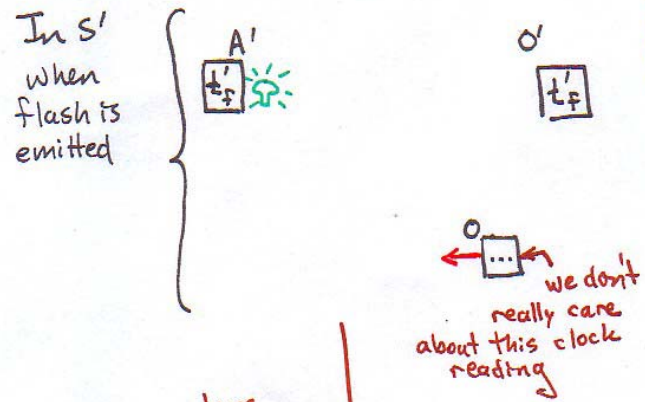
$$t'_f = t'_c - \frac{L_r}{c} = \frac{L_r}{\gamma^2(c-V)} - \frac{L_r}{c} = \frac{L_r}{c} \left( \frac{1 - \frac{V^2}{c^2}}{1 - \frac{V}{c}} - 1 \right) = \frac{L_r}{c} \left( \frac{\left(1 + \frac{V}{c}\right)\left(1 - \frac{V}{c}\right)}{1 - \frac{V}{c}} - 1 \right) = \frac{L_r}{c} \left( 1 + \frac{V}{c} - 1 \right) = \frac{L_r V}{c^2} \quad .$$

Applying this result to the top picture in the right column, we see that indeed, as seen from S, the "chasing clock" (i.e. the clock of A') leads by  $L_r V / c^2$ .

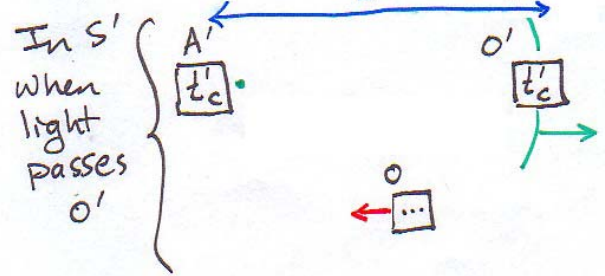
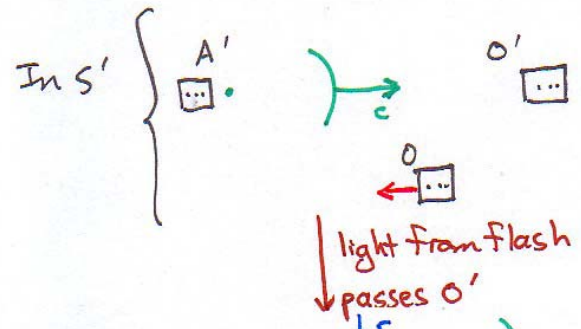
### As Observed in $S'$



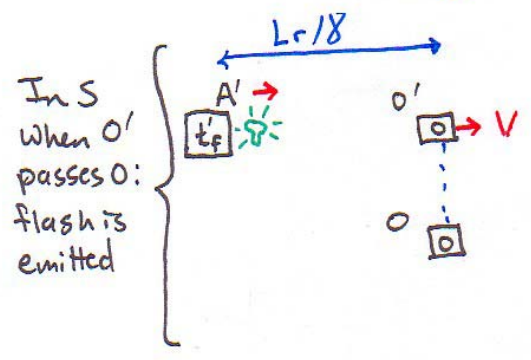
time  $t'_f$  passes



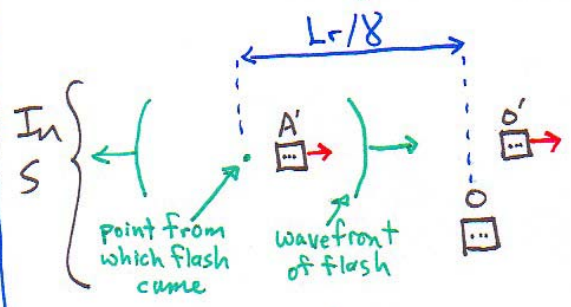
time passes  
light expands out at speed  $c$



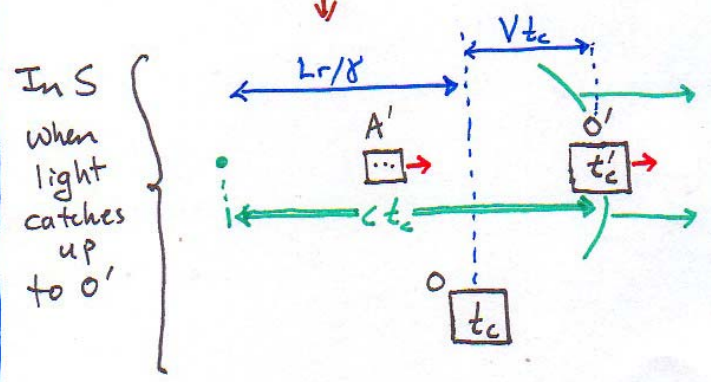
### As Observed in $S$



time passes  
light expands out at speed  $c$  from the point where the flash occurred



time passes  
light from flash catches up with  $O'$



## Spacetime Coordinate Transformations

(There's a good Star Trek phrase for you!) It can be challenging to correctly take all four effects of relativity into effect for a given problem. It would be much easier if we could have a simple set of equations, which, given the spacetime coordinates (*i.e.* given  $x$ ,  $y$ ,  $z$ , and  $t$ ) for an event (such as a flashbulb going off) in one reference frame would allow us to easily find the coordinates in any other reference frame. This set of equations is called the "Lorentz transformation"/

## The Galilean Transformation

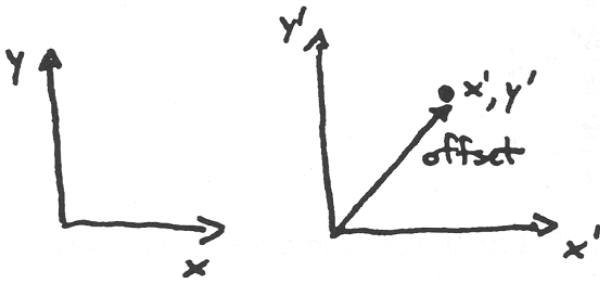
For now, let's return to the intuitive arena where things move at speeds much less than  $c$ , and examine how we can express coordinates of an event as measured in  $S$  in terms of the coordinates as measured in  $S'$ . By "event" I mean simply something with well-defined  $x$ ,  $y$  and  $z$  coordinates and a well-defined time when it happens. Good examples include an explosion, a collision, and a particular tick on a particular clock.

Let's say an event occurs at coordinates  $x'$ ,  $y'$ ,  $z'$ , and  $t'$  in the  $S'$  reference frame. (Again,  $S'$  moves to the right with speed  $V$  relative to  $S$ .) What are the coordinates  $x$ ,  $y$ ,  $z$  and  $t$  of the event as measured by the observers in  $S$ ? We assume as we will for the remainder of our treatment of relativity that the origins ( $x = y = z = 0$ ) of the two reference frames coincide at  $t = t' = 0$  (as measured by clocks at the origins). We lose no generality by doing this, since we can always choose where  $t = 0$ . Because of the synchronization, we immediately have that  $t = t'$ .

Let's start with an easy case:  $x' = y' = z' = 0$ , *i.e.*, the event occurs at the origin of  $S'$  at time  $t'$ . Since  $S'$  moves to the right with speed  $V$ , at time  $t = t'$ , the  $S'$  origin is at

$$x = Vt = Vt', \quad y = 0, \quad z = 0.$$

Now consider an event that occurs someplace else in  $S'$ , at coordinates  $x'$ ,  $y'$ ,  $z'$ ,  $t'$ . This is really just like the case we just considered, except now there is an offset relative to the  $S'$  origin of  $x'$ ,  $y'$ ,  $z'$ :



So, the coordinates in  $S$  are offset by the same amount, *i.e.*,

$$\begin{aligned} x &= x' + Vt' \\ y &= y' \\ z &= z' \\ t &= t' \end{aligned}$$

These relations between the event coordinates  $x'$ ,  $y'$ ,  $z'$ ,  $t'$  as measured on  $S'$  and those as measured on  $S$  is called the "Galilean transformation." It's really nothing new, but just a formal way of writing what you already understand about things moving at relatively small speeds.

## The Galilean Velocity Transformation

A simple consequence of the Galilean transformation is the velocity addition rule which you're used to, as we'll show here. We consider not just a single event which occurs at  $x'$ ,  $y'$ ,  $z'$ ,  $t'$ , but rather an object which is moving, *i.e.*, its coordinates  $x'$ ,  $y'$ , and  $z'$  depend on time. The components of its velocity, as measured in the

$S'$  frame, are found as usual by taking the derivatives with respect to time:  $u'_x \equiv \frac{dx'}{dt'}$   $u'_y \equiv \frac{dy'}{dt'}$   $u'_z \equiv \frac{dz'}{dt'}$ . By taking the derivative of the Galilean transformation with respect to  $t'$ , we can find the relationship between these velocities (measured in  $S'$ ) and those measured in  $S$ :

$$\left. \begin{array}{l} x = x' + Vt' \\ y = y' \\ z = z' \end{array} \right\} \xrightarrow{\frac{d}{dt'}} \left\{ \begin{array}{l} \frac{dx}{dt} = \frac{dx'}{dt'} + V = u'_x + V \\ \frac{dy}{dt} = \frac{dy'}{dt'} = u'_y \\ \frac{dz}{dt} = \frac{dz'}{dt'} = u'_z \end{array} \right.$$

Since  $t = t'$ , we have that  $dt = dt'$ , so  $u_x \equiv \frac{dx}{dt} = \frac{dx'}{dt'}$ , etc. Substituting this into the above gives the Galilean velocity transformation:

$$\begin{aligned} u_x &= u'_x + V \\ u_y &= u'_y \\ u_z &= u'_z \end{aligned}$$

This should make intuitive sense to you, but as we've just discussed, it doesn't work when the speeds involved are close to  $c$ .

## The Lorentz Transformation

On the next assignment, you will use the four basic effects of relativity to show that the fully correct version of the Galilean transformation (the one that is correct both at low speeds and at speeds close to  $c$ ) is

$$\boxed{\begin{array}{l} x = \gamma(x' + V t') \\ y = y' \\ z = z' \\ t = \gamma\left(t' - x'V / c^2\right) \end{array}}$$

This is called the "Lorentz Transformation".

## The Inverse Lorentz Transformation

It is easy to find the inverse transform:  $S$  and  $S'$  are completely symmetrical, except that, in  $S'$  the frame  $S$  is traveling with velocity  $-V \Rightarrow$  all we do is interchange primed and unprimed variable, and substitute  $-V$  for  $V$ :

$$\Rightarrow \boxed{\begin{array}{l} x' = \gamma(x - Vt) \\ y' = y \\ z' = z \\ t' = \gamma(t - xV / c^2) \end{array}}$$

## Recovery of Galilean Transformation at Low Speeds

We know that the Galilean transformation works quite well when things are moving at reasonable speeds, so we need to show that the Lorentz transformation is consistent with this.

Recall that  $\gamma = \frac{1}{\sqrt{1 - V^2/c^2}}$ . For  $V \ll c$ , this becomes  $\gamma = \frac{1}{\sqrt{1 - 0}} = 1$ . So, for  $V \ll c$ , the Lorentz transform reduces to:

$$\begin{array}{rcl}
 x = \gamma(x' + Vt') & & x \cong x' + Vt' \\
 y = y' & \xrightarrow{\gamma \approx 1} & y = y' \\
 z = z' & & z = z' \\
 t = \gamma\left(t' + x' \frac{V}{c^2}\right) & & t \cong t' + \underbrace{x' \frac{V}{c^2}}_{\approx 0 \text{ since } \frac{V}{c} \approx 0}
 \end{array}$$

So, we indeed recover the Galilean transformation. In other words, the Galilean transformation, as in all of Newtonian physics, is a special case of the Lorentz transformation.