Taylor Series Expansions and Approximations

The Taylor series is mainly used for approximating functions when one can identify a small parameter. Expansion techniques are useful for many applications in physics, sometimes in unexpected ways, as the following example will show. The integral

$$I(\varepsilon) = \int_0^{\varepsilon} \frac{x^3 dx}{e^x-1}$$

comes up in the Debye theory of the heat capacity of solids (in Statistical Physics.) The integral cannot be found in tables unless $\varepsilon \to \infty$. Suppose one is interested in the small ε case. Then x never gets large and we can expand the integral and integrate term by term. Table 1 following gives all the formulas we'll need. For small x

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$e^x - 1 = x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots$$

so $$\frac{x^3}{e^x-1} = \frac{x^2}{1 + (\frac{1}{2}x + \frac{1}{6}x^2 + \cdots)}$$

$$= x^2 (1 + y)^n$$

where $y = \frac{1}{2} x + \frac{1}{6} x^2 + \cdots$ is small, if x is, and $n = -1$.

$$(1 + y)^n = 1 + ny + \frac{n(n-1)}{2} y^2 + \cdots$$

$$(1+y)^{-1} = 1 - y + y^2 + \cdots$$

Thus $$\left(1 + (\tfrac{1}{2}\,x + \tfrac{1}{6}\,x^3)\right)^{-1} = 1 - (\tfrac{1}{2}\,x + \tfrac{1}{6}\,x^2) + (\tfrac{1}{2}\,x + \tfrac{1}{6}\,x^2)^2 + \cdots$$

$$= 1 - \frac{1}{2}\,x - \frac{1}{6}\,x^2 + (\frac{1}{4}\,x^2 + \frac{1}{6}\,x^3 + \frac{1}{36}\,x^4) + \cdots$$

$$= 1 - \frac{1}{2}\,x + \frac{1}{12}\,x^2 + \cdots$$

Note that it would be inconsistent to keep the terms of order x^3 and x^4 because their coefficients would be altered by the contributions from y^3 and y^4.

Finally we're ready to integrate

$$I(\varepsilon) = \int_o^{\varepsilon} x^2 \left(1 - \frac{1}{2}\,x + \frac{1}{12}\,x^2 + \cdots\right) dx$$

$$= \int_o^{\varepsilon} \left(x^2 - \frac{1}{2}\,x^3 + \frac{1}{12}\,x^4 + \cdots\right) dx$$

$$= \frac{1}{3}\,\varepsilon^3 - \frac{1}{8}\,\varepsilon^4 + \frac{1}{60}\,\varepsilon^5 + \cdots$$

This expression is likely to be very accurate for $\varepsilon << 1$. In the theory of solids ε is proportional to the temperature, so the above result implies that for low temperatures, the energy of a solid is proportional to the cube of T.

Table 1 lists other useful expansions.

Table 1

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$$

$$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \ldots \text{ for } x^2 < 1$$

$$= \frac{\pi}{2} - \frac{1}{x} + \frac{1}{3x^3} - \frac{1}{5x^5} + \ldots \quad \text{for } x^2 > 1$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

$$a^x = 1 + x \ln a + \frac{(x \ln a)^2}{2!} + \frac{(x \ln a)^3}{3!} + \ldots$$

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots$$

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots$$

$$\ln (1 \pm x) = \pm x - \frac{x^2}{2} \pm \frac{x^3}{3} - \frac{x^4}{4} \pm \ldots$$

$$\sin^{-1} x = x + \left(\frac{1}{2}\right)\left(\frac{x^3}{3}\right) + \left(\frac{1}{2}\right)\left(\frac{3}{4}\right)\left(\frac{x^5}{5}\right) + \left(\frac{1}{2}\right)\left(\frac{3}{4}\right)\left(\frac{5}{6}\right)\left(\frac{x^7}{7}\right) + \ldots$$

$$\tan^{-1} x = x - \frac{1}{3x} + \frac{1}{5x^5} - \frac{1}{7x^7} + \ldots \quad \text{for } x^2 < 1$$

$$\ln x = 2\left[\frac{x-1}{x+1} + \left(\frac{1}{3}\right)\left(\frac{x-1}{x+1}\right)^3 + \left(\frac{1}{5}\right)\left(\frac{x-1}{x+1}\right)^5 + \ldots\right] \quad \text{for } x > 0$$

$$\sec^2 x = 1 + x^2 + \left(\frac{2}{3}\right)x^4 + \ldots$$

$$\ln(\cos x) = -\frac{x^2}{2} - \frac{x^4}{12} - \frac{x^6}{45} - \ldots$$

$$\ln(1 + \sin x) = x - \frac{1}{2}x^2 + \frac{1}{6}x^3 - \frac{1}{12}x^4 + \ldots$$

$$\ln(1 + e^x) = \ln 2 + \frac{1}{2}x + \frac{1}{8}x^2 - \frac{1}{192}x^4 + \ldots$$

$$(a + b)^n = a^n + \frac{n}{1!}\left(a^{n-1}\right)b + \frac{n(n-1)}{2!}\left(a^{n-2}\right)b^2 + \ldots$$

$$(1 + x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \ldots$$

These approximations derive from expressing the function (call it f(x) for originality) in the form of the following power series, the Maclaurin series:

$$\sum_{j=0}^{\infty} \frac{1}{j!} x^j f^{(j)} \Big|_{x=0}, \tag{1}$$

where $f^{(j)}$ denotes the jth derivative of f(x), evaluated in this case at x=0. The series expansion for f(x) = cos x, for example, would be expressed as:

$$\cos x = \left(\frac{1}{0!}\right)(\cos 0)\left(x^0\right) + \left(\frac{1}{1!}\right)\left[\frac{d}{dx}(\cos 0)\right]\left(x^1\right) + \left(\frac{1}{2!}\right)\left[\frac{d^2}{dx^2}(\cos 0)\right]\left(x^2\right) + \ldots$$

$$= (1)(1)(1) + 1(-\sin 0)\,x + \left(\frac{1}{2}\right)(-\cos 0)\,x^2 + \ldots \tag{2}$$

$$= 1 + 0 - \left(\frac{1}{2}\right)x^2 + \ldots = 1 - \left(\frac{1}{2}\right)x^2 + \ldots$$

The Taylor series, which is the more general case of the Maclaurin series, involves expanding about a point “a”, instead of about zero as in the Maclaurin series. The form of the Taylor series is simply:

$$f(x) = \sum_{j=0}^{\infty} \frac{1}{j!} (x - a)^{j} f^{(j)} \Big|_{x=a} , \tag{3}$$

f(x) = sin x thus could be expanded about x = 3 as:

$$\begin{aligned} \sin x &= \left(\frac{1}{0!}\right)(\sin 3)(x - 3)^{0} + \left(\frac{1}{1!}\right)(\cos 3)(x - 3)^{1} + \left(\frac{1}{2!}\right)(\sin 3)(x - 3)^{2} + \ldots \\ &= \sin 3 + (\cos 3)(x - 3) - \left(\frac{1}{2}\right)(\sin 3)(x - 3)^{2} + \ldots \end{aligned} \tag{4}$$

These series expansions are useful under the following conditions.
If f(x) has derivatives of all orders (up to n, with no gaps, n being the number of terms you wish to include in the approximation), on an open interval I, then having picked a point “a” in I, you can make the nth degree Taylor polynomial

$$P_n(x) = \sum_{j=0}^{\infty} \frac{1}{j!}(x - a)^{j} f^{(j)} \Big|_{x=a} \tag{5}$$

$$= f(a) + f'(a)(x - a) + \frac{1}{2} f''(a)\,(x - a)^{2} + \ldots + \frac{1}{n!} f^{(n)}(a)(x - a)^{n} . \tag{6}$$

The difference between the infinite Taylor series and the Taylor polynomial with n terms is the remainder

$$R_n(x) = f(x) - P_n(x) . \tag{7}$$

As long as $R_n(x)$ goes to zero as n goes to infinity, or in other words, if the approximation gets better with higher degree polynomials, then you can safely approximate f(x) with the Taylor series expansion $P_n(x)$.

Another way of stating the condition is that $P_n(x)$ must converge. You can test to see whether a power series converges, and over what interval, by using the Ratio Test. There are other tests for convergence, but the Ratio Test is fairly simple, and applicable in our case.

Ratio Test:

Given the series

$$f(x) = \sum_{i=1}^{\infty} b_i x_i, \tag{8}$$

its infinite limit is:

$$\rho = \lim_{j \to \infty} \left| \frac{b_{n+1} x^{n+1}}{b_n x^n} \right| . \tag{9}$$

Then, if $\rho < 1$, the series converges.

if $\rho > 1$, the series diverges.

if $\rho = 1$, no conclusions can be drawn - series may converge or diverge.

In the case that $\rho < 1$, then the length of the interval over which the series converges is 2R, R being the radius of convergence. The interval of convergence is $x = a \pm R$. R is not really explicitly calculated, but having found ρ, R is the range of x's for which $\rho < 1$. If $\rho = \infty$, then the series always diverges. If $\rho = 0$, then it converges for all x. And if ρ is some function of x, then R is the range of x's which will cause the series to converge.

Example:

Find the interval of convergence for ln (1 + x).

The Maclaurin series expansion yields:

$$\ln(1 + x) = x + \frac{x^2}{2} - \frac{x^3}{3} - \ldots + (-1)^n \left(\frac{x^n}{n} \right) + \ldots \tag{10}$$

Applying the ratio test,

$$\lim_{n \to \infty} \left| \frac{b_{n+1} x^{n+1}}{b_n x^n} \right| = \lim_{n \to \infty} \left| \frac{nx}{n+1} \right| = \lim_{n \to \infty} \frac{n}{n+1} |x| = |x| . \tag{11}$$

Thus, $\rho < 1$ if $|x| < 1$ series converges for $-1 < x < 1$.

Second Example:

Expand $f(x) = \sqrt{x}$ about $x = 1$, using Taylor's series (expansion is valid for the interval $0 < x < 2$ only).

$f(x) = \sqrt{x}$, $f(1) = 1$, $f'(1) = \frac{1}{2}$, $f''(1) = -\frac{1}{4}$, and $f'''(1) = \frac{3}{8}$.

$$P_3(x) = 1 + \frac{1}{2}(x - 1) - \frac{1}{2!}\,\frac{1}{4}(x - 1)^2 + \frac{1}{3!}\,\frac{3}{8}(x - 1)^3 \qquad (12)$$

$$= 1 + \frac{1}{2}(x - 1) - \frac{1}{8}(x - 1)^2 + \frac{1}{16}(x - 1)^3 .$$

Third Example:

Evaluate approximately by series expansion around $x = 0$

$$\int_0^{\pi/2} \frac{\sin x}{x}\, dx .$$

Using the expansion of sin x.

$$\int_0^{\pi/2} \frac{\sin x}{x}\, dx = \int_0^{\pi/2} \frac{1}{x}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots \right) dx$$

$$= \int_0^{\pi/2} \left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \dots \right) dx$$

$$= \left(x - \frac{x^3}{18} + \frac{x^5}{600} - \dots \right) \tfrac{1}{2}{}_{0}^{\pi/2}$$

$$= \frac{\pi}{2}\left(1 - \frac{\pi^2}{72} + \frac{\pi^4}{9600} - \dots \right) = 1.371 .$$

References: Shenk, Calculus and Analytic Geometry
Guterman and Nitecki, Differential Equations
Swartz, Used Math
Stephenson, Worked Examples in Mathematics for Scientists and Engineers