a theory in a 3D context is necessary. A decisive test of theories would be to see whether incorporating the interaction of the sidewall in various experimental geometries can produce the observed general trend of the velocity field (shear banding). [2]

In the preceding discussions, the curvature of the circular channel in our system and other Coutte-cell experiments are generally ignored. Circular geometries are advantageous for eliminating "end effects" and are commonly used in both 2D and 3D experiments. But in view of the long-range "force chains" (as can be visualized by the use of photo-elastic materials as in Ref. [24]), the finite radius of curvature may still play a minor role in the flow, especially for quasi-statically packed grains.

5.3.3 Development of bulk properties with packing thickness

Several notable bulk properties observed in our system occur only if the packing thickness is sufficient:

Distinct stats of order – In the experiments of our typical particles with $d = 0.68$mm, we find that about 15 layers of materials or more are needed for developing two clearly distinguishable states of order. The states of internal order for experiments with less amount of glass beads depend on the exact condition of the bottom surface: the degree of order can be highly sensitive to layer thickness (Fig.3.4), or can be inhomogenous around the channel (with the shear force close to the typical value for a disordered state. For packings thicker than 30 layers, spatially non-uniform states with a detectable vertical gradient of local order can be created and persist indefinitely. (Please see Section 3.2, Section 3.3, and Fig. 5.3 for these observation.) Therefore, the "available phase space" of the system in terms of its state of order depends on system size; our typical packing size (24 layers with W_0/d= 28.5) lies within the range that a simple two-state scenario is sufficient.

[2] In units of particle diameter, the spatial decay of velocity field produced in existing 2D theories and simulations such as Ref. [59, 1] generally have a length scale an order of magnitude larger than all experimental measurements reviewed here.

Anomalous mobility in response to reversals of boundary motion – The "soft period" upon the reversal of shearing, i.e. the anomalous mobility during the shear-reversal transients requires at least 13 layers in thickness for the phenomena to develop fully (Section 5.2.3).

Deviation from a linear velocity profile (Shear banding) – We demonstrate in Fig. 5.4 that the vertical velocity profile progressively deviates from a linear profile (for 5-layer flow) to a non-linear master curve (for 12 layers or more). This behavior may involve not only the sufficient development of bulk property, but also the crossover of relative importance of the sidewalls and the rough bottom.

5.3.4 Rate-independent frictional dynamics

The grains in the entire sheared packing are in a quasi-static regime, as pointed out in Appendix A.1. In this regime where particles are in contact with multiple neighbors, friction is expected to be an important parameters of the problem. Comparing behaviors of dry particles and fluid-immersed particles (Fig. 3.3) suggests that changing the microscopic friction substantially changes the internal velocity fields of particles: without the lubrication introduced by the immersion fluid, the precursor before the crystallization transition is found to be an order of magnitude longer (under the same driving speed and boundary conditions). On the other hand, we find that the lubrication introduced by the fluid reduces the shear stress at the driving boundary only by about 12 percent (Sec. 5.1.4), under a fixed normal load. This observation implies that the shear stress detected at the driving boundary is transmitted mainly through the particle-scale surface irregularities, rather than microscopic friction forces that are tangent to the contact area.

Over the entire two decades of driving speeds, we do not find properties that are dependent on shear rate; this is consistent with the notion that the sheared grains are completely under the quasi-static regime. We do not find rate-dependent phase transition or 'shear-melting' as in some other systems, which often reflect the competition between the rate (or energy) of imposed shearing and the rate (or energy) of local activity associated with bulk excitation (such as mechanical vibration or thermal motion). At the level of compression stress imposed by the normal load (minimally (1.3Kg)g), the entire range of applied shear rate is too small (such that $X << 1$ in Appendix A.1) compared to the minimum that is required for any individual grain to be temporarily independent of its neighbors.

As a result, the underlying physics of the dynamics of sheared quasi-static packing studied here is perhaps mainly about the *adjustment* of a packed structure driven under boundary constraints, as opposed to reflecting instantaneous competitions between different factors. The quasi-static adjustments are rate-independent and often irreversible. In Section 5.2.1, we illustrate the rate-independent adjustment of internal grains to a new direction of flow, in the form of an anomalous mobility during the shear-reversal transient. A fully non-linear model (that does not require a finite linear regime) may be necessary for describing this behavior. Interestingly, densely packed grains may adjust to alternating shearing with a fixed amplitude, as well as to a stationary shearing in a fixed direction - see the experiments reported in Ref. [42] as an example, where grains show stepwise descents in heights only when the amplitude of imposed cyclic shearing is abruptly change.

5.3.5 Granular flows vs. Fluid flows

How does the velocity field of a slowly sheared granular packing compare to a viscous fluid driven in the same geometry?

In Appendix A.2 we discuss the consequence of the fact that, in the simple case of an incompressible Newtonian fluid , the velocity field for the steady flow satisfies a 2D Laplace equation $\nabla^2 v_x = 0$ across a rectangular channel. This equation couples the second derivatives (the "curvatures") in two orthogonal directions. The vertical decay lengths are well coupled to the horizontal wave numbers that that are selected by the finite-width channel due to the no-slip boundary conditions. The principal decay length is therefore $1/\lambda_1 = (W_0/n\pi)|_{n=1} = W_0/\pi$. (See the slope on the semi-log plot Fig. 5.5). Note that this is exact only when the fluid is Newtonian. Otherwise, the more general stress balance (as in Eqn. A.7) requires

$$(\nabla\eta) \cdot (\nabla v_x) + \eta \nabla^2 v_x = 0 \tag{5.1}$$

on the yz plane rather than $\nabla^2 v_x = 0$. Here, one can see that the two "curvatures" of the velocity field are comparable ($\frac{\partial^2 v_x}{\partial z^2} \approx -\frac{\partial^2 v_x}{\partial y^2}$) when the non-Newtonian term of the