Proof that $\psi(\varphi=0)=\psi(\varphi=2\pi)$

$\hat{L}_z$ is Hermitian $\Rightarrow$ for arbitrary state $|\chi\rangle$, we have

χ chi, not x

$$\langle\psi|\hat{L}_z|\chi\rangle = \langle\chi|\hat{L}_z|\psi\rangle^*$$

express in $\vec{r}$-basis, but only show φ-dependence explicitly

$$\underbrace{\int_0^{2\pi}\psi^*(\varphi)\frac{\hbar}{i}\frac{\partial}{\partial\varphi}\chi(\varphi)\,d\varphi}_{\text{LHS}} = \left[\int_0^{2\pi}\chi^*(\varphi)\frac{\hbar}{i}\frac{\partial}{\partial\varphi}\psi(\varphi)\,d\varphi\right]^*$$

Integrate by parts: $v\equiv\chi \Rightarrow dv=\frac{d\chi}{d\varphi}d\varphi$

$u\equiv\psi^*$

$$\Rightarrow \text{LHS} = \frac{\hbar}{i}\left[\psi^*\chi\Big|_0^{2\pi} - \int_0^{2\pi}\chi\frac{d\psi^*}{d\varphi}d\varphi\right] = -\frac{\hbar}{i}\int_0^{2\pi}\chi\frac{d\psi^*}{d\varphi}d\varphi$$

$$\Rightarrow \psi^*\chi\Big|_0^{2\pi}=0.$$

Since ψ and χ are arbitrary, we must have $\psi^*(\varphi=2\pi)=\psi^*(\varphi=0)$

take adjoint

$$\psi(\varphi=2\pi)=\psi(\varphi=0)$$