Making the S-matrix clear

Walter F. Smith, Bryn Mawr College 2-24-10

There is still some confusion about the interpretation of the matrix elements of the S-matrix, so let me attempt to clarify it.

This is a language problem. One of the applications of the S-matrix is to convert the matrix representation of an operator from one basis to another. For example:

$\hat{A}_x = S^{\dagger}\hat{A}_z\, S$

where $\hat{A}_x$ means the matrix form of $\hat{A}$ in the *x*-basis, and $\hat{A}_z$ means the matrix form of $\hat{A}$ in the *z*-basis. In this case, the matrix elements in S are of the form $\langle +z|+x\rangle$. In this case, we might say that the matrix elements have the form <old basis of operator | new basis of operator>, since <u>in the whole process</u> we're converting $\hat{A}$ from the *z*-basis to the *x*-basis.

However, we can use that very same S-matrix for the following operation:

$$\psi_z = S\psi_x \;,$$

where $\psi_x = \begin{pmatrix} \langle +z|\psi\rangle \\ \langle -z|\psi\rangle \end{pmatrix}$ is the column matrix form of $|\psi\rangle$ in the *z*-basis, and $\psi_x = \begin{pmatrix} \langle +x|\psi\rangle \\ \langle -x|\psi\rangle \end{pmatrix}$ is the column matrix form of $|\psi\rangle$ in the *x*-basis. In this case, since S is converting $|\psi\rangle$ from the *x*-basis to the *z*-basis, it would be more appropriate to say that the matrix elements have the form <final basis|original basis>, even though they are still of the form $\langle +z|+x\rangle$.

In both cases, we're really using the S-matrix itself to accomplish the same task: converting from the *x*-basis to the *z*-basis. When we're converting the operator $\hat{A}$ from the *z*-basis to the *x*-basis, the final result needs to be something that can operate on vectors in the *x*-basis. In other words, we want

$$\hat{A}_x\psi_x = \varphi_x \,, \qquad (1)$$

where φ_x is the result of using the operator $\hat{A}$ on $|\psi\rangle$, expressed in the *x*-basis. Let's see how this works when we write $\hat{A}_x$ as $S^{\dagger}\hat{A}_z\, S$, and apply it to ψ_x:

$$\hat{A}_x\psi_x = S^{\dagger}\hat{A}_z\, S\psi_x$$

The first thing that happens on the right side is $S\psi_x$. The S-matrix converts the ψ_x to the z-basis, so $S\psi_x = \psi_z$. Then, the $\hat{A}$ operation takes place. Since $|\psi\rangle$ is now expressed in the *z*-basis, the *z*-basis version of $\hat{A}$ works great: $\hat{A}_z\psi_z = \varphi_z$, where φ_z is the result of using the operator $\hat{A}$ on $|\psi\rangle$, expressed in the *z*-basis. Finally, the $S^{\dagger}$ matrix changes from the *z*-basis to the *x*-basis, so $S^{\dagger}\varphi_z = \varphi_x$, as desired (see equation (1)).

In an effort to sum up: the matrix elements of the S-matrix are best thought of as

<final basis|original basis>, since that is the function of the S-matrix by itself. When it is used together with the $S^{\dagger}$ matrix, as in $S^{\dagger}\hat{A}_z S$, the process converts a matrix from one basis to another, and one could for that application think of the matrix elements of S as having the form <old basis of operator | new basis of operator>. In both applications, the bras and kets of the S and $S^{\dagger}$ matrices always point toward their kind:

$$\hat{A}_x = S^{\dagger}\hat{A}_z S$$

$$\hat{A}_x = \begin{pmatrix} \langle +x|+z\rangle & \langle +x|-z\rangle \\ \langle -x|+z\rangle & \langle -x|-z\rangle \end{pmatrix} \hat{A}_z \begin{pmatrix} \langle +z|+x\rangle & \langle +z|-x\rangle \\ \langle -z|+x\rangle & \langle -z|-x\rangle \end{pmatrix}$$

Note how the bras and kets that contain *z* point toward the $\hat{A}_z$.

Similarly,

$$\psi_z = S\psi_x$$

$$\psi_z = \begin{pmatrix} \langle +z|+x\rangle & \langle +z|-x\rangle \\ \langle -z|+x\rangle & \langle -z|-x\rangle \end{pmatrix} \psi_x$$

Note how the bras, which contain *z*, point toward the ψ_z, and the kets, which contain *x*, point toward the ψ_x.