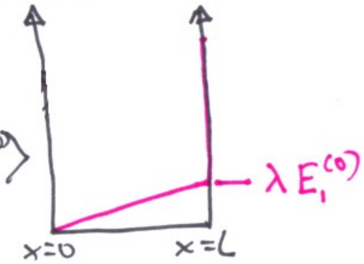
## Example of non-degenerate perturbation theory:

The $\infty$ square well

$$\hat{H}_0 = \frac{\hat{p}^2}{2m} + V_0 \qquad \hat{H}_0 |\varphi_n^{(0)}\rangle = E_n^{(0)} |\varphi_n^{(0)}\rangle$$

$$V_0 = \begin{cases} 0 & \text{inside} \\ \infty & \text{outside} \end{cases}$$



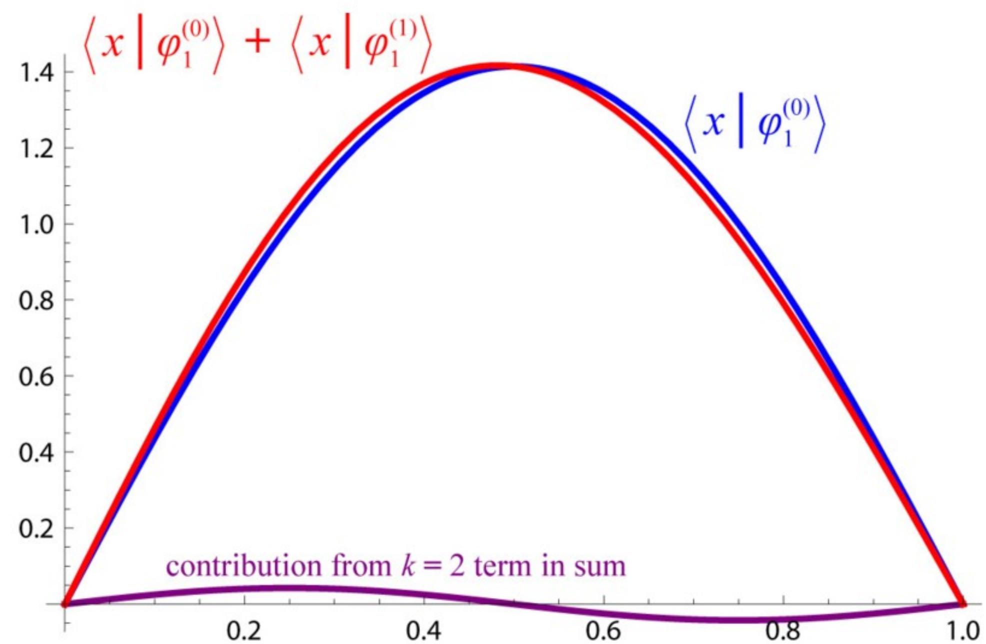

Perturbation: $\lambda \hat{H}_1 = \lambda E_1^{(0)} \frac{\hat{x}}{L}$

$$\langle x|\psi_n\rangle \cong \underbrace{\langle x|\varphi_n^{(0)}\rangle}_{\sqrt{\frac{2}{L}}\sin\frac{n\pi x}{L}} + \frac{\lambda E_1^{(0)}}{L} \sum_{k \neq n} \underbrace{\langle x|\varphi_k^{(0)}\rangle}_{\sqrt{\frac{2}{L}}\sin\frac{k\pi x}{L}} \frac{\langle \varphi_k^{(0)}|\hat{x}|\varphi_n^{(0)}\rangle}{E_n^{(0)} - E_k^{(0)}}$$

$$E_n^{(0)} = \frac{n^2 h^2}{8mL^2}$$

$$\rightarrow \langle x|\psi_1\rangle \cong \sqrt{\frac{2}{L}} \sin\frac{\pi x}{L} + \sqrt{\frac{2}{L}} \frac{8\lambda}{\pi^2} \sum_{k=2,4,\ldots} k \frac{\sin k\pi x/L}{(k^2-1)^3}$$

For $\lambda = 0.5$ (larger than we'd normally use for perturbation theory):



## Degenerate perturbation theory

Example: $|\varphi_{n,1}^{(0)}\rangle$ & $|\varphi_{n,2}^{(0)}\rangle$ are eigenstates of $\hat{H}_0$, both with eigenvalue $E_n^{(0)}$.

We seek to find the two eigenstates of $\hat{H} = \hat{H}_0 + \lambda \hat{H}_1$ such that

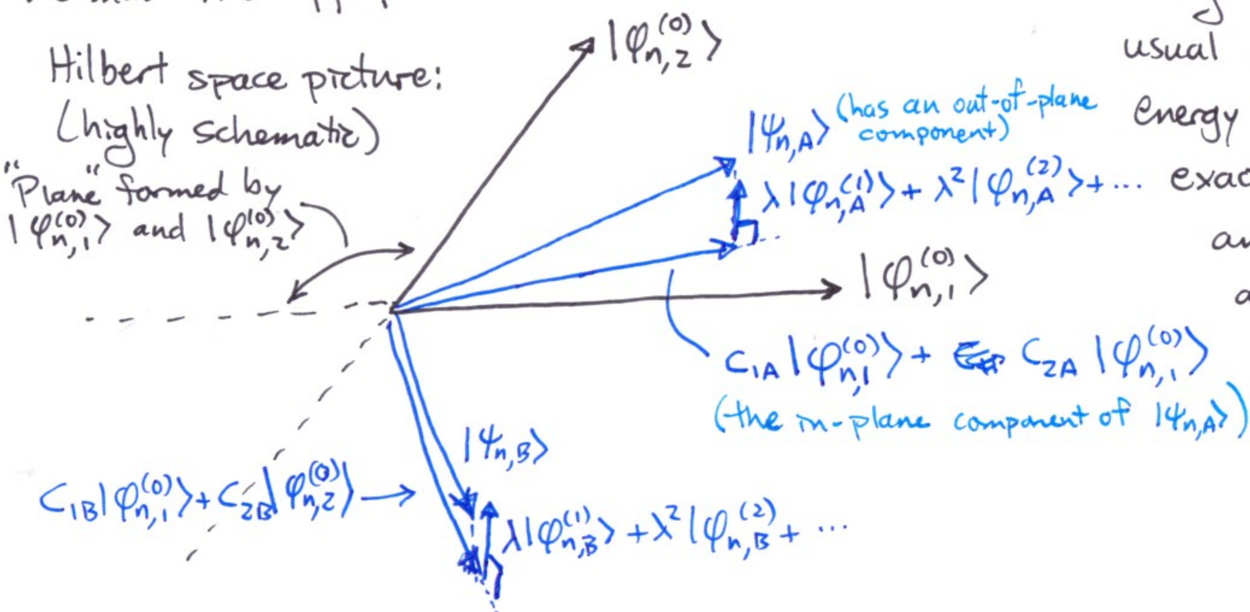
$$\hat{H}|\psi_{n,A}\rangle = E_{n,A}|\psi_{n,A}\rangle$$

$$\hat{H}|\psi_{n,B}\rangle = E_{n,B}|\psi_{n,B}\rangle$$

and $\lim\limits_{\lambda\to 0} E_{n,A} = \lim\limits_{\lambda\to 0} E_{n,B} = E_n^{(0)}$.

In general, neither $|\varphi_{n,1}^{(0)}\rangle$ nor $|\varphi_{n,2}^{(0)}\rangle$ is a zeroth-order approximation for $|\psi_{n,A}\rangle$ or $|\psi_{n,B}\rangle$. Instead, we must find appropriate linear combinations:

Hilbert space picture: (highly schematic)




$$|\psi_{n,A}\rangle = \sum_i c_{iA}|\varphi_{n,i}^{(0)}\rangle + \lambda|\varphi_{n,A}^{(1)}\rangle + \dots$$

$$E_{n,A} = E_n^{(0)} + \lambda E_{n,A}^{(1)} + \dots$$

$$|\psi_{n,B}\rangle = \sum_i c_{iB}|\varphi_{n,i}^{(0)}\rangle + \lambda|\varphi_{n,B}^{(1)}\rangle + \dots$$

$$E_{n,B} = E_n^{(0)} + \lambda E_{n,B}^{(1)} + \dots$$

Following Townsend, we now suppress the A and B subscripts

$$\to \begin{pmatrix} \langle\varphi_{n,1}^{(0)}|\hat{H}_1|\varphi_{n,1}^{(0)}\rangle & \langle\varphi_{n,1}^{(0)}|\hat{H}_1|\varphi_{n,2}^{(0)}\rangle & \cdots \\ \langle\varphi_{n,2}^{(0)}|\hat{H}_1|\varphi_{n,1}^{(0)}\rangle & \langle\varphi_{n,2}^{(0)}|\hat{H}_1|\varphi_{n,2}^{(0)}\rangle & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \end{pmatrix} = E_n^{(1)} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \end{pmatrix}$$

Solving this eigenvalue equation by the usual methods gives the first-order energy shifts $E_n^{(1)}$ for each of the exact eigenstates $|\psi_{n,A}\rangle$, $|\psi_{n,B}\rangle \dots$, and also gives the zeroth-order approximation for the exact state:

$$|\psi_n\rangle \simeq \sum_i c_i|\varphi_{n,i}^{(0)}\rangle$$