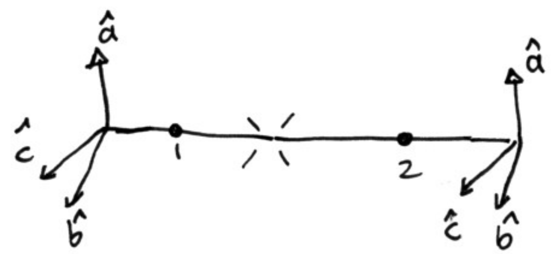
John Bell (1964): What if the two observers make measurements along three non-orthogonal axes?



The two observers use the same set of three axes. Now the hidden variables must be ready for 8 possibilities. In each case, if the two observers make measurements along the same axis, they must get opposite results, to conserve angular momentum.

Probability	Particle 1 $\hat{a}$ $\hat{b}$ $\hat{c}$	Particle 2 $\hat{a}$ $\hat{b}$ $\hat{c}$
N_1	+ + +	− − −
N_2	+ + −	− − +
N_3	+ − +	− + −
N_4	+ − −	− + +
N_5	− + +	+ − −
N_6	− + −	+ − +
N_7	− − +	+ + −
N_8	− − −	+ + +

Since probabilities are positive,

$$N_3 + N_4 \le (N_2 + N_4) + (N_3 + N_7)$$

Probability of measuring $S_{a1} = \frac{\hbar}{2}$ and $S_{b2} = \frac{\hbar}{2}$

$$\equiv P(+\hat{a}, +\hat{b})$$

From the chart, $N_3 + N_4 = P(+\hat{a}, +\hat{b})$

$N_2 + N_4 = P(+\hat{a}, +\hat{c})$ $\quad N_3 + N_7 = P(+\hat{c}, +\hat{b})$

$\Rightarrow$ Hidden variable theory predicts

$$\hat{P}(+\hat{a}, +\hat{b}) \le P(+\hat{a}, +\hat{c}) + P(+\hat{c}, +\hat{b})$$

a "Bell's inequality".

What does conventional quantum mechanics predict?

$P(+\hat{a}, +\hat{b}) = |\langle +\hat{a}, +\hat{b} | 0, 0 \rangle|^2$

Let's choose $\theta_{ab} = 120°$, $\theta_{ac} = 60° = \theta_{cb}$

$$\rightarrow P(+\hat{a}, +\hat{b}) = \tfrac{3}{8} > \underbrace{P(+\hat{a}, +\hat{c})}_{1/8} + \underbrace{P(+\hat{c}, +\hat{b})}_{1/8}$$

Thus, this prediction for the direction of the inequality is opposite that of hidden variable theory.

1982: Aspect et al. perform analogous experiments on photons

$\Rightarrow$ hidden variable theory is wrong!

The position operator

$|x\rangle$: a position eigenstate, with the particle completely localized at position x

$$\hat{x}\,|x\rangle = x\,|x\rangle$$

Because x varies continuously, we must modify the identity operator:

$$\hat{I} = \sum_i |i\rangle\langle i| \longrightarrow \hat{I} = \int_{-\infty}^{\infty} dx\, |x\rangle\langle x|$$

$\Rightarrow |\psi\rangle = \int_{-\infty}^{\infty} dx\, |x\rangle\langle x|\psi\rangle$ a generic state, expressed as a superposition of position eigenstates

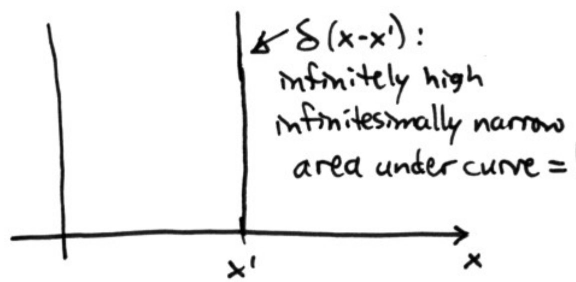
*** $\boxed{\psi(x) \equiv \langle x|\psi\rangle}$ ***

The wavefunction

special case: $|\psi\rangle = |x'\rangle \Rightarrow |x'\rangle = \int_{-\infty}^{\infty} dx\, |x\rangle\langle x|x'\rangle$

$\Rightarrow \langle x|x'\rangle = \delta(x-x')$, the "Dirac delta function"

$$\int_{-\infty}^{\infty} f(x)\,\delta(x-x')\,dx = f(x')$$



Normalization

$$\langle\psi|\psi\rangle = 1 \dashrightarrow \boxed{\int_{-\infty}^{\infty} |\psi|^2 dx = 1}$$

Expectation values

$\hat{A}(x)$: an operator that can be computed based on knowledge of x

$$\dashrightarrow \boxed{\langle A(x)\rangle = \int_{-\infty}^{\infty} |\psi|^2 A(x)\, dx}$$

The translation operator

$\hat{T}(a)|x\rangle = |x+a\rangle$, i.e. the state is translated to the right by a

The generator of translations

$$\hat{T}(dx) = 1 - \frac{i}{\hbar}\hat{g}\,dx$$

$\hat{g}$ ← generator of translations

Townsend pp. 153-5 $\rightarrow [\hat{x}, \hat{g}] = i\hbar$

The momentum operator

The "correspondence principle" states that quantum mechanics must give predictions that are consistent with classical mechanics, when applied to large objects.

$$\Rightarrow \langle p_x \rangle = m \frac{d\langle x \rangle}{dt}$$

$$\rightarrow [\hat{x}, \hat{p}_x] = i\hbar$$

Therefore, we identify the generator of translations $\hat{g}$ with the momentum operator $\hat{p}_x$.