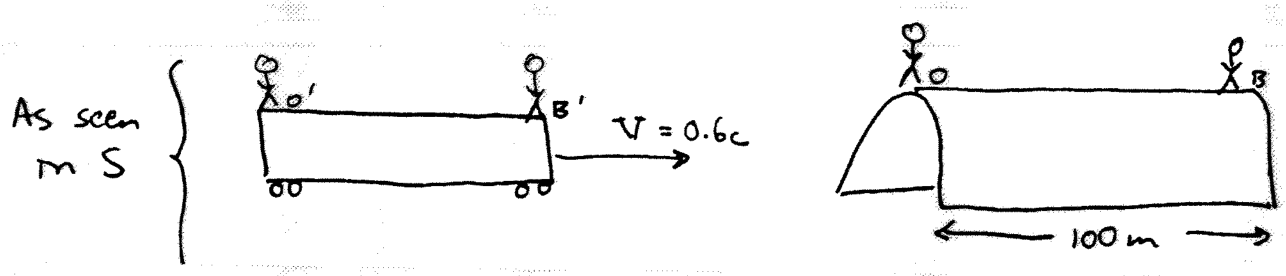
FAMOUS PARADOXES

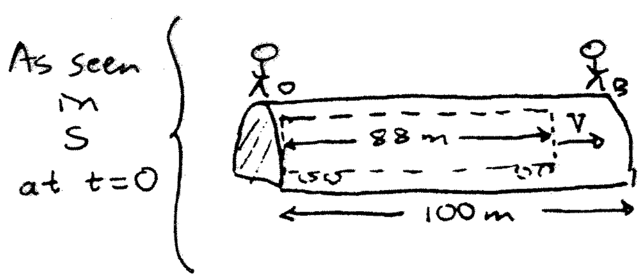
The train and tunnel paradox. A train of length 110 m (as measured by observers on the train) approaches a tunnel at speed V = 0.6c. The tunnel has a length of 100 m, as measured by stationary observers:



Let's call the reference frame of the tunnel S and that of the trains S'. The people in S' measure the proper length of the train ⇒ the length of the train in S is $L_{other} = L_T/\gamma$

$$\gamma \equiv \frac{1}{\sqrt{1-V^2/c^2}} = 1.25 \Rightarrow L_{other} = \frac{110\,m}{1.25} = 88\,m$$

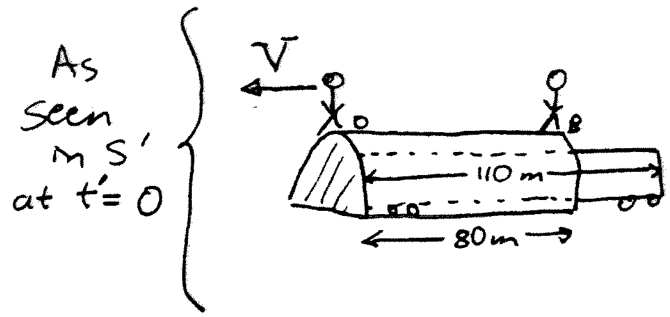
Thus, to people in S, the train is shorter than the tunnel. Observers O and B plan to briefly "catch" the train inside the tunnel. At the instant that O' passes O (call this t = t' = 0, as usual), both O and B will briefly slam down gates at each end of the tunnel, catching the train inside:



To avoid bloodshed, O and B will quickly re-open their doors after slamming them.

However, the people in S' see things very differently. The people in S measure the proper length of the tunnel ⇒the length of tunnel in S' is $L_{other} = L_p/\gamma = \frac{100\text{m}}{1.25} = 80$ m.

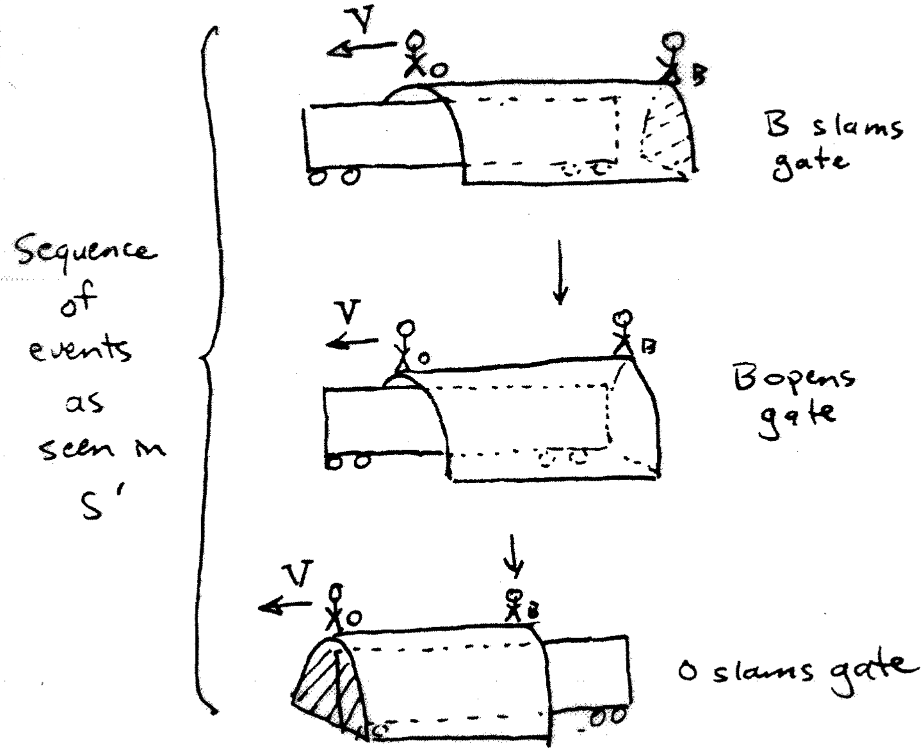
Since their train is 110 m long, there is no way that it can be "caught" in a tunnel that is only 80 m long. They agree that O slams her door at the instant O' passes, however, their view of the situation is:



So, who is right? Does the train get "caught" or not? The answer is that both are right, and whether or not the train is caught depends on your reference frame!

Qualitative Explanation

In S', O and B are moving to the left ⇒ B has the chasing clock ⇒ his clock leads that of O. Thus, at the instant B slams his gate (when his clock reads t = 0), the clock at O does not yet read t = 0, and O has not yet slammed her door. B then opens his door, the tunnel keeps moving to the left, and then O slams her gate:



Quantitative Treatment

Let x = 0 be the position of O and x' = 0 be the position (in S') of O'. Then O slams her gate at x = x' = 0 and t = t' = 0. What is the time and place (in S') that B slams his gate?

We'll use the Lorentz transform (in the inverse version):

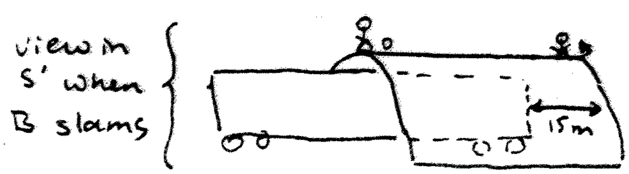
$$x' = \gamma\ (x - Vt)$$
$$t' = \gamma\ (t - xV/c^2)$$

The position of B is x = 100 in. (since the tunnel is 100 m long in S) and he slams his gate simultaneously with O, i.e., at t = 0

$\Rightarrow$ the position of B when he slams his gate is

$$x' = \gamma\ [100\ \text{m} - V\ (0)]\ = 125\ \text{m}$$

Since the position of B' is x' = 110 m (since the train is 110 m long), this means that B slams his gate 15 m in front of the train:



The time (in S') when B slams is

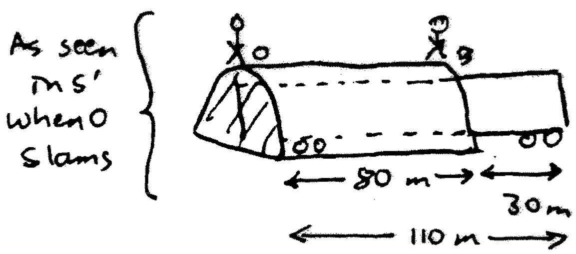
$$t' = \gamma\ (t - x\,V/c^2) = 1.25\ (0 - 100\text{ m}\ (0.6\text{ c})/c^2)$$
$$= -2.5_{-7}\ \text{s} = -0.25\ \mu\text{ s}$$

Since O slams her gate at t' = 0, this means that B slams his gate 0.25 μ s before O does (as seen in S').

During this 0.25 μ s interval, the tunnel moves a distance of

$$0.25\ \mu\text{ s x } 0.6\text{c} = 45\text{ m} \Rightarrow \text{the train now protrudes by } 45\text{ m} - 15\text{ m} = 30\text{ m}$$

So, the situation when O slams is:



The Twin Paradox. Twin α stays on earth and sees twin β departing at V = 0.6c. Ten years later β returns, having spent half the trip traveling outbound at 0.6c, turning around upon reaching planet X, and then spending the second half of the trip returning to earth at 0.6c. As we know, α says that β will age more slowly, so that β will age less than α during the trip. However, β also says that α ages more slowly, so we might expect that β would say that α will age less during the trip. However, once β returns, we can compare them side by side, and we'd see that in fact β is the one who is younger. Doesn't this indicate that the reference frame of α is "preferred," in violation of the basic postulate of relativity?

Qualitative Explanation. Because β had to accelerate to turn around, he didn't stay in an inertial reference frame for the whole trip, while α did. Therefore the situation really isn't symmetrical, and this is why β ages less.

Quantitative Treatment.

To analyze this correctly, we need three reference frames:

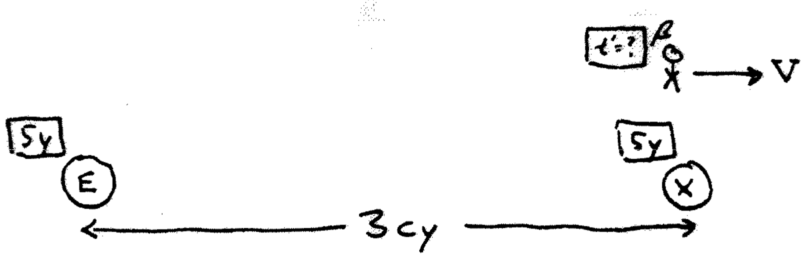
Earth $\leftrightarrow$ S

$V = 0.6$ c $\leftrightarrow$ S' (Twin β for first half of trip)

$V = -0.6$c $\leftrightarrow$ S" (Twin β for second half)

In S: during the first 5y ("year"), B travels a distance (5y) (0.6c) = 3 cy ("lightyears")

So, as seen in S, the situation when β reaches planet X is:



(Displays of digital clocks shown in boxes)

What does the clock of β read when he reaches planet X?

We use the inverse Lorentz transform:

$$t' = \gamma(t - xV/c^2) = 1.25\left[5y - (3c\cancel{y})(0.6\cancel{c})/\cancel{c^2}\right]$$
$$= 1.25[5y - 1.8y] = 4y$$

In other words, at this moment β says, "Only 4 y have passed. You people of planet X have mis-synchronized your clock with the one on earth. Although I can't see him, I know that his clock runs slow, so I think he's aged $\frac{4\ y}{\gamma}$ = 3.2y during my trip."

We can check the logic of β using the Lorentz transform to find the reading on the clock of α at $t' = 4$ y. Since we know x for α but not his position in S', it's easiest to use

$$t' = \gamma(t - xV/c^2) \qquad t = \frac{t'}{\gamma} + \frac{xV}{c^2} \qquad (1)$$

$\longrightarrow$ $x = 0$ for $\alpha \Rightarrow t = \dfrac{t'}{\gamma} = \dfrac{4\ y}{1.25} = 3.2y$ (in agreement with the logic of β).

Now (when he reaches planet X), β jumps from S' to S". When he arrives in S" the clock on planet X still reads 5 y (as it did just before his jump). However, the clocks in S" near planet X read

$$t'' = \gamma(t - xV/c^2) = 1.25[5y - (3cy)(-0.6\cancel{c})/\cancel{c^2}]$$
$$= 1.25[5y + 1.8y] = 8.5y$$

The new comrades of β in S" inform him that the clock of α now reads (using (1) above)

$$t = \frac{t''}{\gamma} + \underbrace{\frac{xV}{c^2}}_{\to 0 \text{ as above}} = \frac{8.5y}{\gamma} = 6.8y$$

This means that, if β accepts the reports of his comrades in S", he must accept that his twin has suddenly aged by 3.6y when he jumped from S' to S". NOTE: No one making local observations on α would see any sudden change in his age.

The return trip of β (in S") again takes 4 y as measured by β. During this trip, α again ages by $\frac{4\ y}{\gamma} = 3.2y$ as measured by observers in S".

$\Rightarrow$ Upon the return of β, he has aged by 4 y + 4 y = 8y.

However, α has aged by $\underbrace{3.2\ y}_{\text{trip out}} + \underbrace{3.6\ y}_{\text{Upon jump from S' to S"}} + \underbrace{3.2\ y}_{\text{trip back}} = 10\ y$ } this is the aging process perceived by β

So, when β returns both twins agree that β has aged 8 y while α has aged 10 y, but they disagree on the details of how this took place.

Reminder: This statement means that in S' the events of 1) β reaching planet X and 2) The clock of α reading are simultaneous. Of course, these events are not simultaneous in S. As seen in S, when β reaches planet X, the clock of α reads 5y, not 3.2y.