Partial Differentiation and Wave Equations

Suppose you have to take the derivative of a function f(x,y,z). How does this differ from taking the derivative of a function g(x)? The change in g, Δg, is related only to the change in one variable, Δx. But as x, y, and z change, f will change.

$$df = \frac{\partial f}{\partial x}\, dx + \frac{\partial f}{\partial y}\, dy + \frac{\partial f}{\partial z}\, dx \qquad (1)$$

What do those strange ∂'s mean? They signify partial differentiation, rather than the ordinary total differentiation that we would do to take the derivative of g(x). To take the partial derivative, we pretend that all the variables except one are constants and take the derivative with respect to that one variable.

For example, let $f(x, y, z) = x^3 z + y^2 + 6xy + 7z$.

$$\frac{\partial f(x,y,z)}{\partial x} = 3x^2 z + 6y \qquad (2)$$

in taking the partial derivative of f(x,y,z) with respect to x, we treat y and z as constants. The other two partial derivatives are

$$\frac{\partial f(x,y,z)}{\partial y} = 2y + 6x \qquad (3)$$

$$\frac{\partial f(x,y,z)}{\partial z} = x^3 + 7$$

One application of partial differention in physics is in solving wave equations. A classical wave is a solution to the partial differential equation

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}\ . \qquad (4)$$

(v is the velocity with which the wave travels.) Let's see how this equation arises for waves on a string. Let y(x,t) denote the displacement of an element of length Δx of the string at a position x and at time t. The mass of this element of string is $m=\rho\Delta x$, where ρ is the linear mass density of the string. Thus Newton's second law applied to the element gives

$$\rho\, \Delta x \frac{\partial^2 y}{\partial t^2} = \mathbf{F}_{net}\ \hat{\mathbf{j}} \qquad (5)$$

where the net force (in the direction perpendicular to the string) is provided by the difference in the tension forces acting on the two ends of the element of string as shown in fig. 1. The force is dotted with a unit vector in eq. (5) to focus on the acceleration in the y-direction.

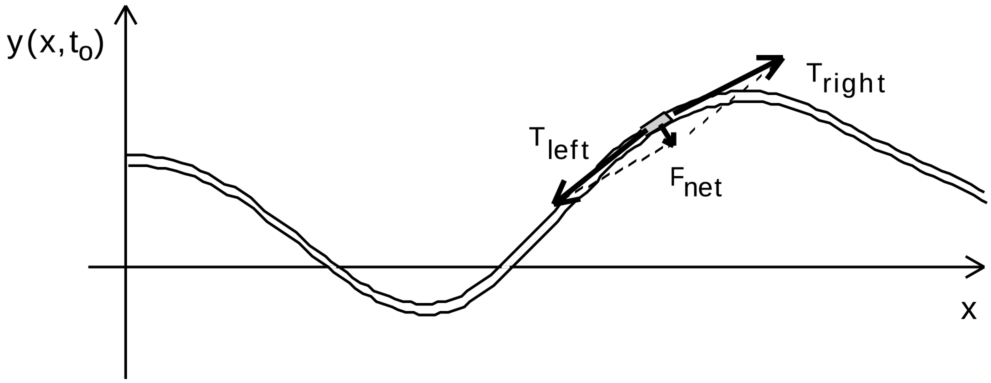

Fig. 1: Tension forces acting on an element of the string add to produce the net force, $\mathbf{F}_{net}$.

The tension forces acting to the left and right are similar in magnitude and would cancel exactly except that the curvature of the string causes their directions not to be perfectly alligned. The net force they exert is therefore proportional to this curvature, to Δx and to the tension magnitude, T.

$$\mathbf{F}_{net} \cdot \hat{\mathbf{j}} = T \frac{\partial^2 y}{\partial x^2} \Delta x \tag{6}$$

so that eq. (4), the wave equation, follows by combining eqs. (5,6).

A solution to equation (4) is

$$y(x,t) = A \sin[k(x - vt)] \tag{7}$$

where A is an arbitrary amplitude and k is the wave vector, $k = 2\pi/\lambda$ (any λ is possible). Calculating the partial derivatives explicitly, we get

$$\frac{\partial^2 y}{\partial x^2} = -k^2 A \sin[k(x - vt)] \tag{8}$$

$$\frac{\partial^2 y}{\partial t^2} = -k^2 v^2 A \sin[k(x - vt)]$$

Clearly eq. (4) is satisfied. Note also that replacing the sin function in eq. (7) with any other function of (x-vt) will still lead to a correct solution. Thus waves on an ideal string can have any shape.

References: All intermediate calculus texts cover partial differentiation

Griffiths, *Introduction to Electrodynamics* Chapter 1