Notation for General Relativity and Other Advanced Topics

The notation used in General Relativity, most of which is Differential Geometry, can be quite confusing at first. This section is not intended to teach any General Relativity, but to familiarize the student with the notation before s/he takes a course in it and is completely swamped and clueless. The tensor notation is also useful in lots of other 300-level courses, especially Particle Physics.

Objects:

We now introduce the notation of μ and ν and company. μ and ν are variables which vary from 0 to 3, i.e. $\mu = 0, 1, 2, 3$. Generally Greek letters when used as indices will vary from 0 to 3 and Roman letters vary from 1 to 3.

4-vectors: A four-vector $\vec{a}$, or a^μ is just a vector with four components instead of the usual three. (We will discuss why we need four shortly.) The type of coordinates doesn't matter much; we can have a 4-vector $\vec{a} = (t, x, y, z)$ or a 4-vector $\vec{a} = (t, r, \theta, \varphi)$. Usually we know what coordinate system we are working in if we care about the components, though, so we generally use the notation $a^\mu = \vec{a} = (a^0, a^1, a^2, a^3)$. These will have dot products with the minus sign on the a^0 term (see the next section, on operations).

one-forms: These are sort of the opposite of 4-vectors. They are notated by $a^\mu = \tilde{a} = (a_0, a_1, a_2, a_3)$. They use contractions (described below).

tensors: Ah, what an evil and frightening word. Not to worry, though, they are not as scary as they look. Tensors are something like n-dimensional matrices. For example, a 4-vector a^μ is also a $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ tensor, and a one-form a_ν is also a $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ tensor. The upper number is called contravariant, also, and the lower number is called covariant. (The upper number in the brackets counts the number of superscripted indices and the lower number counts the number of subscripted indices.) We can make a tensor go up in rank, too. A $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$ tensor, also notated by something like $T^{\alpha\beta}$ or a $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$ tensor like $T_{\alpha\beta}$ is a

4-by-4 matrix: $\begin{pmatrix} a_{00} & a_{01} & a_{02} & a_{03} \\ a_{10} & a_{11} & a_{12} & a_{13} \\ a_{20} & a_{21} & a_{22} & a_{23} \\ a_{30} & a_{31} & a_{32} & a_{33} \end{pmatrix}$. Now, for larger ranks of tensors, we can mentally imagine the matrix having more dimensions, first like a cube, then a hypercube, and so on. A $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ tensor would be notated $R^{\alpha\beta}{}_{\gamma}$.

Operations:

"**inner product**": In normal math and physics, we have a metric on our space. A metric is basically a way of measuring distances; for regular three dimensional stuff, the metric is $ds^2 = dx^2 + dy^2 + dz^2$. We have the usual inner product on this space (also called the dot product and notated that way), which is $\vec{a} \quad \vec{b} = (a_1, a_2, a_3) \quad (b_1, b_2, b_3) = a_1b_1 + a_2b_2 + a_3b_3$. Now, in General Relativity, we have a fourth dimension, time, and it goes into the metric as a negative so we have $ds^2 = dx^2 + dy^2 + dz^2 - dt^2$. Thus, when we take a dot product (this isn't actually an inner product but instead a semi-inner product) we do it as $\vec{a} \quad \vec{b} = (a_0, a_1, a_2, a_3) \quad (b_0, b_1, b_2, b_3) = -a_0b_0 + a_1b_1 + a_2b_2 + a_3b_3$. We always subscript the time component with 0.

contraction: This is just the real, genuine inner product (yes, all pluses) but you don't use it for vectors. You use it for one-forms, which will be described in the next section. Sometimes it looks like $\tilde{p}(\vec{A}) = A^{\alpha}p_{\alpha} = A^0p_0 + A^1p_1 + A^2p_2 + A^3p_3$.

Summation Notation:

When you have one index up and another down, notated by the same letter, then it is assumed that you will sum over them. In other words, $\Lambda^{\mu}{}_{\nu}A^{\nu}$ really means $\sum_{\nu=0}^{3} \Lambda^{\mu}{}_{\nu}A^{\nu}$.

Other Notation:

$g(\mathbf{A},\mathbf{B}) = \mathbf{A} \quad \mathbf{B} = \eta_{\mu\nu}A^{\mu}B^{\nu}$

g^{00} (or g_{00}) is the time coefficient in any metric.

τ is proper time and l_0 is proper length.

$\tilde{e}$ or $\vec{e}$ are basis one-forms or vectors.

$T^{\alpha\beta}{}_{,\beta}$ means to take partial derivatives with respect to the β variables.

It can be kind of confusing to see notation like G_{rr} or $G_{\varphi\varphi}$ instead of, say, G_{11} or G_{33}. But $G_{r\varphi}$, for example, simply means to take whatever is in the φ position of the r row. Usually this is the same as G_{13}.

u^μ is the 4-velocity and $u^\mu = \gamma(1,\vec{v})$, where γ has the usual special relativistic meaning.

p is pressure and $p^\mu = (E, p^k)$.

$R^\alpha{}_{\beta\gamma\delta}$ is the Riemann Curvature Tensor.

$R^{\alpha\beta}$ is the Ricci Tensor and R is the Ricci Scalar. (You get these by contracting the Riemann Curvature Tensor in various ways.)

The Einstein Tensor is $G^{\alpha\beta} = R^{\alpha\beta} - \frac{1}{2}g^{\alpha\beta}R$.

Another notation to look out for is the Affine Connection Coefficient. It's related to finding "straight lines" in curved spaces, and looks like

$$\Gamma_{\alpha}{}^{\nu}{}_{\beta} = \frac{\partial^2 x^\mu}{\partial x^\alpha \partial x^\beta}\frac{\partial x^\nu}{\partial x^\mu} = \frac{1}{2}g^{\nu\mu}(g_{\mu\alpha,\beta} + g_{\mu\beta,\alpha} - g_{\alpha\beta,\mu})$$

Reference: Steve Boughn's Fall 1989 General Relativity notes.