# Matrices

A matrix is an efficient way to represent or manipulate a set of elements, such as the coefficients of simultaneous equations. In using them, one may have to multiply two or more matrices, and it isn't immediately obvious how matrix multiplication is done.

The following matrices are given as examples:

matrix **A** $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ matrix **B** $\begin{pmatrix} e & f \\ g & h \end{pmatrix}$

matrix **C** $\begin{pmatrix} k \\ l \end{pmatrix}$ matrix **D** $\begin{pmatrix} m & n \end{pmatrix}$ matrix **E** $\begin{pmatrix} o & p \\ q & r \end{pmatrix}$

In order to find the matrix **F** which is the product of any two of the above matrices, one multiplies the elements in each row of the first matrix by the corresponding elements in each column of the second matrix. The sum of the products at the end of each row is an element of **F**. For example,

$$\mathbf{F} = \mathbf{A}\,\mathbf{B} = \begin{pmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{pmatrix}. \qquad (1)$$

Formally stated,

$$f_{ij} = \sum_{k} a_{ik}\, b_{kj}\,, \qquad (2)$$

where i = row number, j = column number, the a's are the elements of **A**, b' s the elements of **B** and f's the elements of **F**. k would run from 1 to the number of columns. For multiplication, the number of elements in each row of **A** must be equal to the number of elements in the columns of **B**. The numbers of rows in **A** and **B** needn't be the same, nor the columns. For example:

$$\mathbf{A\,C} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} k \\ l \end{pmatrix} = \begin{pmatrix} ak + bl \\ ck + dl \end{pmatrix}$$

$$\mathbf{D\,B} = \begin{pmatrix} m & n \end{pmatrix}\begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} me + ng & mf + nh \end{pmatrix}$$

$$\mathbf{A\,D} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} m & n \end{pmatrix} \text{ doesn't work!}$$

Matrices larger than 2 x 2 follow the same rules.

**Example:**

The Pauli spin matrices, which describe a spin-1/2 particle, are:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{3}$$

We can show that $\sigma_i^2 = 1$. ( 1 is commonly used for the matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ .)

$$\sigma_1^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0*0 + 1*1 & 0*0 + 1*0 \\ 1*0 + 0*1 & 1*1 + 0*0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\sigma_2^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} -i^2 & 0 \\ 0 & -i^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\sigma_3^2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & (-1)^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Matrix multiplication is not usually commutative, i.e., $\mathbf{A\,B} \neq \mathbf{B\,A}$ . However, it is associative: $(\mathbf{A\,B})\,\mathbf{C} = \mathbf{A}\,(\mathbf{B\,C})$ , and distributive: $\mathbf{A}\,(\mathbf{B} + \mathbf{E}) = \mathbf{A\,B} + \mathbf{A\,E}$.

For matrix **addition**, the numbers of rows and columns, respectively, for each matrix must be the same.

Here, we give brief descriptions of various types of matrices, just for familiarity's sake. Longer discussions of what they are and what they're used for can be found in Arfken or Boas.

**Diagonalized Matrix:**

A diagonalized matrix is one in which all elements not on the main diagonal are zero. Matrices can be diagonalized using row reduction (see appropriate section).

**Inverse Matrices:**

A matrix **A** is invertible if there exists a matrix $\mathbf{A}^{-1}$ such that $\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$ (the identity matrix). The only matrices which are invertible are those $n \times n$ matrices with nonzero determinant.

**Transpose of a Matrix:**

The transpose of a matrix is the matrix you get by switching the rows and columns of a matrix, i. e. if $a_{ij}$ is the (i,j)th entry of **A**, $a_{ji}$ is the (i,j)th entry of $\mathbf{A}^T$, or if $\mathbf{A} = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$ then $\mathbf{A}^T = \begin{pmatrix} a & d & g \\ b & e & h \\ c & f & i \end{pmatrix}$ .

**Hermitian conjugate of a matrix:**

The Hermitian conjugate of a matrix is the transpose of its complex conjugate, or in other symbols, $\mathbf{H}^t = (\mathbf{H}^*)^T$ .

**Unitary Matrices:**

A unitary matrix is one whose Hermitian conjugate equals its inverse, notated as $\mathbf{U}^t = \mathbf{U}^{-1}$.

References: Arfken, Mathematical Methods for Physicists

Stephenson, Worked Examples in Mathematics for Scientists and Engineers

Charles Curtis's introductory linear algebra text

See also: Bamberg & Sternberg, A Course in Mathematics for Students of Physics, Vol. I.