L'Hopital’s Rule

Using L'Hopital's Rule allows us to evaluate the limit of a term or an indeterminate function, e.g. , goes to 0/0. For example, we take the limit

$$\lim_{x \to 0} \frac{\sin 6x}{\sin x} . \tag{1}$$

Both the numerator and denominator tend to zero, so we can't tell if the limit equals zero, infinity, or something in between. By L'Hopital's Rule, we take the limit of the derivatives of top and bottom. This yields

$$\lim_{x \to 0} \frac{6 \cos 6x}{\cos x} = 6 . \tag{2}$$

Or, take the case

$$\lim_{x \to \infty} \frac{e^x}{x} = \frac{\infty}{\infty} . \tag{3}$$

This would become

$$\lim_{x \to \infty} \frac{e^x}{1} = e^x = \infty . \tag{4}$$

This second case is one where we would often use an informal, intuitive application of L’Hopital’s Rule. Looking at it, we know that e^x goes to infinity much faster than x . In other words, we’re estimating the slopes mentally instead of formally taking the derivatives. By the time x tends to infinity, the numerator is infinitely larger than the denominator; therefore, the limit = infinity.

Types of Indeterminate Forms

Listed below are the different types of indeterminate forms, which require different tricks to be able to apply L'Hopital's Rule. These should be kept in mind, because it's easy to assume, for example, that something of the form $\infty - \infty$ = zero. And if you apply L'Hopital's Rule to something not indeterminate, you will most likely get the wrong answer.

Table 1

Type of Indeterminate Form:	**Example:**	**Method:**
$\frac{0}{0}, \frac{\pm\infty}{\pm\infty}$	$\lim_{x \to 0} \frac{x}{x^2}$	Take derivative of numerator and denominator.
$(0)(\infty)$	$\lim_{x \to \infty} x \ln x$	Make into type $\frac{0}{0}$ or $\frac{\infty}{\infty}$. I.e., make f(x) g(x) into $\frac{f(x)}{1/g(x)}$.
$0^0, 1^\infty, \infty^0$	$\lim_{x \to 0} x^x$	Take logarithm of expression. I.e., $\ln[f(x)^{g(x)}] = g(x)[\ln f(x)]$ This is now type (0) (∞).
$\infty - \infty$	$\lim_{x \to \infty} \frac{1}{x} - \frac{1}{x^3}$	Fiddle with it algebraically to make it one of the above three types.

Note : should limit of $\frac{\frac{d}{dx}[f(x)]}{\frac{d}{dx}[g(x)]}$ also be indeterminate, apply L'Hopital's Rule agai

Example

Evaluate

$$\lim_{x \to \pi/4} \frac{\cos^2(2x)}{(x - \frac{\pi}{4})^2} .$$

We find that

$$\lim_{x \to \pi/4} \frac{\cos^2(2x)}{(x - \frac{\pi}{4})^2} = \lim_{x \to \pi/4} \frac{-4 \cos 2x \sin 2x}{2(x - \frac{\pi}{4})}$$

$$= \lim_{x \to \pi/4} \frac{-\sin 4x}{x - \frac{\pi}{4}} \qquad (5)$$

$$= \lim_{x \to \pi/4} -4 \cos 4x$$

$$= 4 .$$

L'Hopital's Rule was applied three times before the limit was not of the form 0/0.

References: Shenk, Calculus and Analytic Geometry
Stephenson, Worked Examples in Mathematics for Scientists and Engineers