Fourier Analysis and Transforms

How to Calculate Fourier Series

Any function that is periodic on the interval $[-\pi, \pi]$ can be expressed as a sum of sines and cosines, like this

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \qquad (1)$$

(To prove that any periodic f(x) can be expressed like this, one must show that the set of functions {1, cos nx, sin nx} is a *complete set*. This topic is covered in the section on Families of Orthogonal Functions, and it is explicitly shown there that the given set of functions is in fact complete.)

Assuming that an expansion like eq. (1) is possible, we would then want to know what the a_n's and b_n's are. The derivation of formulas for a_n and b_n is a bit tedious, but it is not difficult and the result is quite simple. Here are the details, but if you just want the results you can skip down to eqs. (4-6).

Derivation of formulas for a_n and b_n

$$f(x) = \frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + \ldots + b_1 \sin x + b_2 \sin 2x + \ldots \qquad (2)$$

Let's multiply through by cos nx on both sides

$$f(x) \cos nx = \frac{a_0}{2} \cos nx + a_1 \cos x \cos nx + \ldots + b_1 \sin x \cos nx + \ldots$$

Next we integrate on both sides of this equation from $-\pi$ to π. The result will be that only one term on the right hand side will survive. As we show in the following, all the rest vanish and this will enable us to deduce an expression for a_n.

$$\int_{-\pi}^{\pi} f(x) \cos nx\, dx = \int_{-\pi}^{\pi} \frac{a_0}{2} \cos nx\, dx + \int_{-\pi}^{\pi} a_1 \cos x \cos nx\, dx + \ldots$$
$$+ \int_{-\pi}^{\pi} b_1 \sin x \cos nx\, dx + \int_{-\pi}^{\pi} b_2 \sin 2x \cos nx\, dx + \ldots \qquad (3)$$

Now we have to do all integrals on the right hand side. The first one is easy.

$$\int_{-\pi}^{\pi} \frac{a_0}{2} \cos nx\, dx = \frac{a_0}{2} \frac{1}{n} \sin nx \Big|_{-\pi}^{\pi} = 0.$$

The ones that have mixed sines and cosines are also easy. They can all be written in the form

$$\int_{-\pi}^{\pi} b_m \sin mx \cos nx \, dx = b_m \int_{-\pi}^{\pi} \frac{1}{2}\left(\sin[(m+n)x] + \sin[(m-n)x] \right) dx$$

where we have used trig identities to reexpress the integrand. Note that if m=n then the second term in the integrand vanishes. It is easy to integrate the simple trig functions of the right hand side so we find

$$\int_{-\pi}^{\pi} b_m \sin mx \cos nx \, dx = -\frac{b_m}{2}\left(\frac{1}{m+n} \cos[(m+n)x] + \frac{1}{m-n} \cos[(m-n)x] \right)\Bigg|_{-\pi}^{\pi}$$

$$= 0 \, .$$

(Both terms evaluate to zero because cos(-θ) = cos(θ), and we do not need to worry about the troublesome case of m=n in the 2nd term because we know that the term is not there if m=n.)

So, we have found so far that the first term of the right hand side of eq. (3) and all of the b_m terms vanish. It only remains to do the integrals that have a product of two cosines, all of which can be expressed as

$$\int_{-\pi}^{\pi} a_m \cos mx \cos nx \, dx = a_m \int_{-\pi}^{\pi} \frac{1}{2}\left(\cos[(m+n)x] + \cos[(m-n)x] \right) dx,$$

where we have again used trig identities to reexpress the product in the integrand. This time if m=n the second term does not vanish, so we'll have to handle that case separately.

Case 1: m≠n

$$\int_{-\pi}^{\pi} a_m \cos mx \cos nx \, dx = \frac{a_m}{2}\left(\frac{1}{m+n} \sin[(m+n)x] + \frac{1}{m-n} \sin[(m-n)x] \right)\Bigg|_{-\pi}^{\pi}$$

$$= 0 \qquad \text{since } \sin n\pi = 0 \text{ for any integer n.}$$

Case 2: m=n

$$\int_{-\pi}^{\pi} a_n \cos nx \cos nx \, dx = a_n \int_{-\pi}^{\pi} \frac{1}{2}\left(\cos 2nx + 1 \right) dx$$

$$= \frac{a_n}{2}\left(\frac{1}{2n} \sin 2nx + x \right)\Bigg|_{-\pi}^{\pi}$$

$$= \pi \, a_n .$$

So, having evaluated all the terms on the right hand side of eq. (4) we have found that all of them vanish except the one that looks like

$$\int_{-\pi}^{\pi} a_n \cos nx \cos nx \, dx$$

and we can therefore say that

$$\int_{-\pi}^{\pi} f(x) \cos nx \, dx = \pi \, a_n,$$

or in other words that

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx \, dx \quad . \tag{4}$$

This whole process can be repeated, multiplying eq. (1) by sin nx and integrating to get

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx \, dx \quad . \tag{5}$$

Finally a_0 is obtained by just integrating eq. (1) from $-\pi$ to π. All the sine and cosine terms integrate to zero over that interval and one is left with

$$\int_{-\pi}^{\pi} f(x) \, dx = \int_{-\pi}^{\pi} \frac{a_0}{2} \, dx = \frac{a_0}{2} \, 2\pi = \pi \, a_0,$$

so that a_0 obeys the same formula as a_n, or

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \, dx \, . \tag{6}$$

Eqs. (4-6) give the desired result, formulas for determining the coefficients of eq. (1) for any arbitrary f(x).

An example

Now that we have the formulas for a_n and b_n we can find the Fourier expansion of any periodic function. Consider for example the ramp function, R(x) sketched in fig. 1.

Let's find the Fourier series for this function. We need to find the a_n's and the b_n's of eq. (1) and can do that using eqs. (4-6). Let's begin with the a_n's for $n \geq 1$.

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} R(x) \cos nx \, dx = \frac{1}{\pi}\int_{0}^{2\pi} R(x) \cos nx \, dx$$

because R(x) is periodic and of course so is cos nx. On the interval $[0,2\pi]$, R(x) is easily specified,

$$R(x) = \frac{x}{2\pi} \, ,$$

so

$$a_n = \frac{1}{\pi}\int_{0}^{2\pi} \frac{x}{2\pi} \cos nx \, dx \, .$$

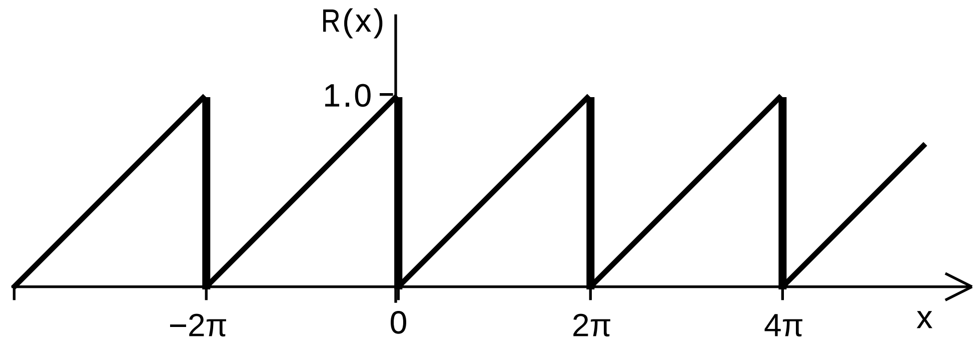

Fig. 1: The ramp function with period 2π.

This integral can be looked up in tables or done by parts (see the section Introduction to Integration). The result is:

$$\int x \cos nx \, dx = \frac{x}{n} \sin nx + \frac{1}{n^2} \cos nx.$$

So

$$a_n = \frac{1}{2\pi^2} \left(\frac{x}{n} \sin nx + \frac{1}{n^2} \cos nx \right) \Bigg|_0^{2\pi} = 0.$$

Next we determine a_0.

$$a_0 = \frac{1}{\pi} \int_0^{2\pi} R(x) \, dx = \frac{1}{\pi} \int_0^{2\pi} \frac{x}{2\pi} \, dx = \frac{1}{2\pi^2} \frac{1}{2} x^2 \Bigg|_0^{2\pi} = 1.$$

Finally we determine the b_n's.

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} R(x) \sin nx \, dx = \frac{1}{\pi} \int_0^{2\pi} R(x) \sin nx \, dx = \frac{1}{\pi} \int_0^{2\pi} \frac{x}{2\pi} \sin nx \, dx .$$

The necessary integration formula for this case is

$$\int x \sin nx \, dx = -\frac{x}{n} \cos nx + \frac{1}{n^2} \sin nx.$$

So

$$b_n = \frac{1}{2\pi^2} \left(-\frac{x}{n} \cos nx + \frac{1}{n^2} \sin nx \right) \Bigg|_0^{2\pi} = \frac{1}{2\pi^2} \left(-\frac{2\pi}{n} \cos 2n\pi + 0 + 0 - 0 \right)$$

$$= -\frac{1}{\pi n} .$$

Assembling all the a_n's and b_n's into eq. (1) gives

$$R(x) = \frac{1}{2} + \sum_{n=1}^{\infty} -\frac{1}{\pi n} \sin nx . \tag{7}$$

This might seem a little unbelievable the first time you see it. How can the straight lines of the ramp function, and those jumps from 1 down to 0 be obtained just by adding up sine functions? To confirm eq. (7) it is helpful to add up the first few terms of the series to see how how the trig functions go about this tricky business. It is easy to do this sort of adding and plotting on a computer. As simple program was used to generate fig. 2 below which shows the sum in eq. (7) terminated after 3 and 6 terms.

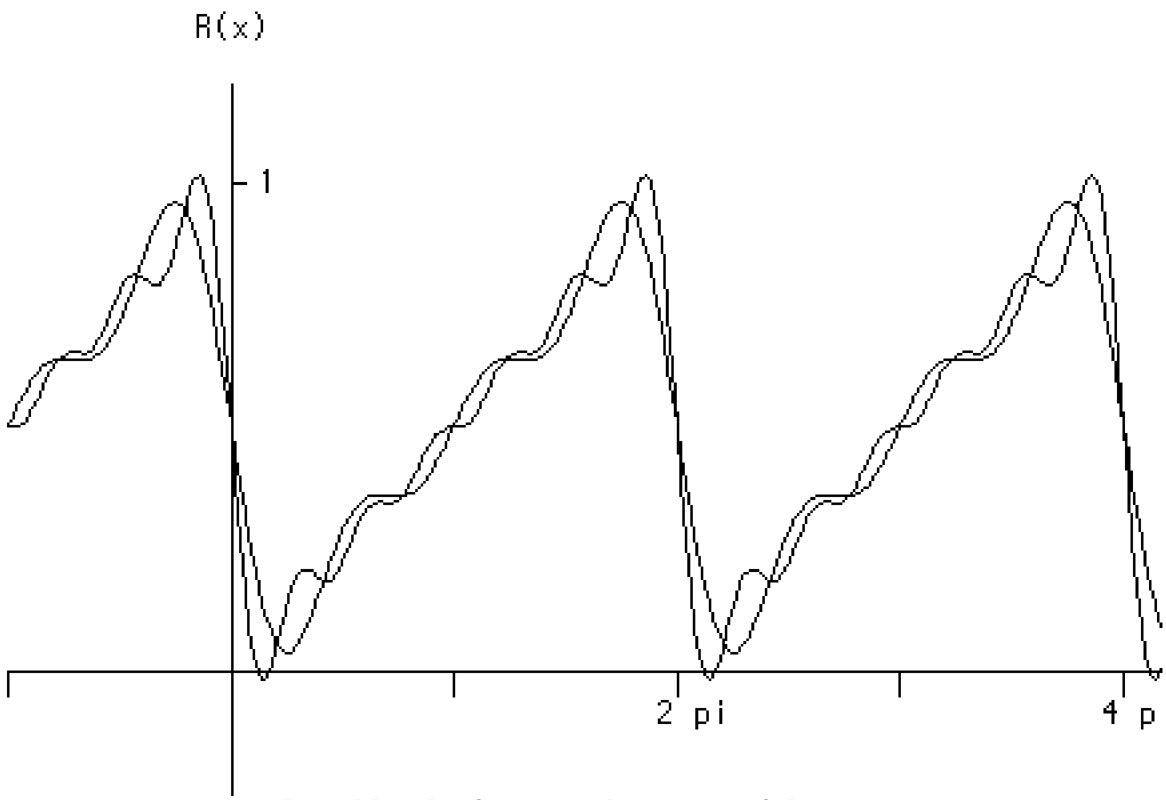

Fig. 2: R(x) as reproduced by the first 3 and 6 terms of the series in eq. (7).

The sum is indeed attempting to reproduce the ramp function rather successfully. Adding further terms will result in further reductions of the remaining wiggles.

What's the point?

Whole books have been written to describe the applications of Fourier analysis in physics. A good recent one is the one by James mentioned at the end of this section. Important applications occur in all areas of physics concerned with waves, especially optics, quantum mechances and the theory of diffraction, and also in other areas concerned with periodic behavior, including (ac) electronics, and vibrations.

One example will show the pattern of these applications. Suppose you want an electrical signal with periodic time dependence like that of the ramp function.[1] Let V(t) be a periodic voltage function with the following specification for $0 \le t < T$

$$V(t) = \frac{V_o}{T}\, t, \qquad (7)$$

where T is the period and V_o is the maximum voltage obtained. V plotted versus t resembles the R(x) of fig. 1.

While it is not possible, using basic circuit components, to produce a voltage like that V(t) directly, it is straightforward to make oscillator circuits that produce pure sinusoidal voltages.[2] Fourier analysis shows us how to combine those sine and cosine type functions to produce any desired periodic function.

To make contact with eq. (1) we can choose $x = \omega_o t$, where $\omega_o = \frac{2\pi}{T}$ is the angular frequency associated with the period, and scale the amplitude via V_o. So

$$V(t) = V_o\left(\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\omega_o t + \sum_{n=1}^{\infty} b_n \sin n\omega_o t\right). \qquad (8)$$

We have already determined the necessary values of the a_n's and b_n's in the example calculation and so find that

$$V(t) = V_o\left(\frac{1}{2} - \sum_{n=1}^{\infty} \frac{1}{\pi n} \sin n\omega_o t\right). \qquad (9)$$

Eq. (9) tells us to combine a dc (time-independent) voltage of $\frac{V_o}{2}$ with sinusoidal voltages of frequency $n\omega_o$ and amplitude $\frac{V_o}{\pi n}$, all phase shifted by 180° to take care of the negative sign in the b_n's.

[1]Perhaps, for example, you need a magnetic field that ramps upward in magnitude repetivitively. In that case you would want a voltage whose time dependence was similar to the x dependence of R(x) to drive the current in a solenoid.

[2]For most simple wave forms we do not need to build our own circuitry. Devices called "function generators", which are designed to implement these sinusoidal additions, are affordably available for use in physics laboratories.

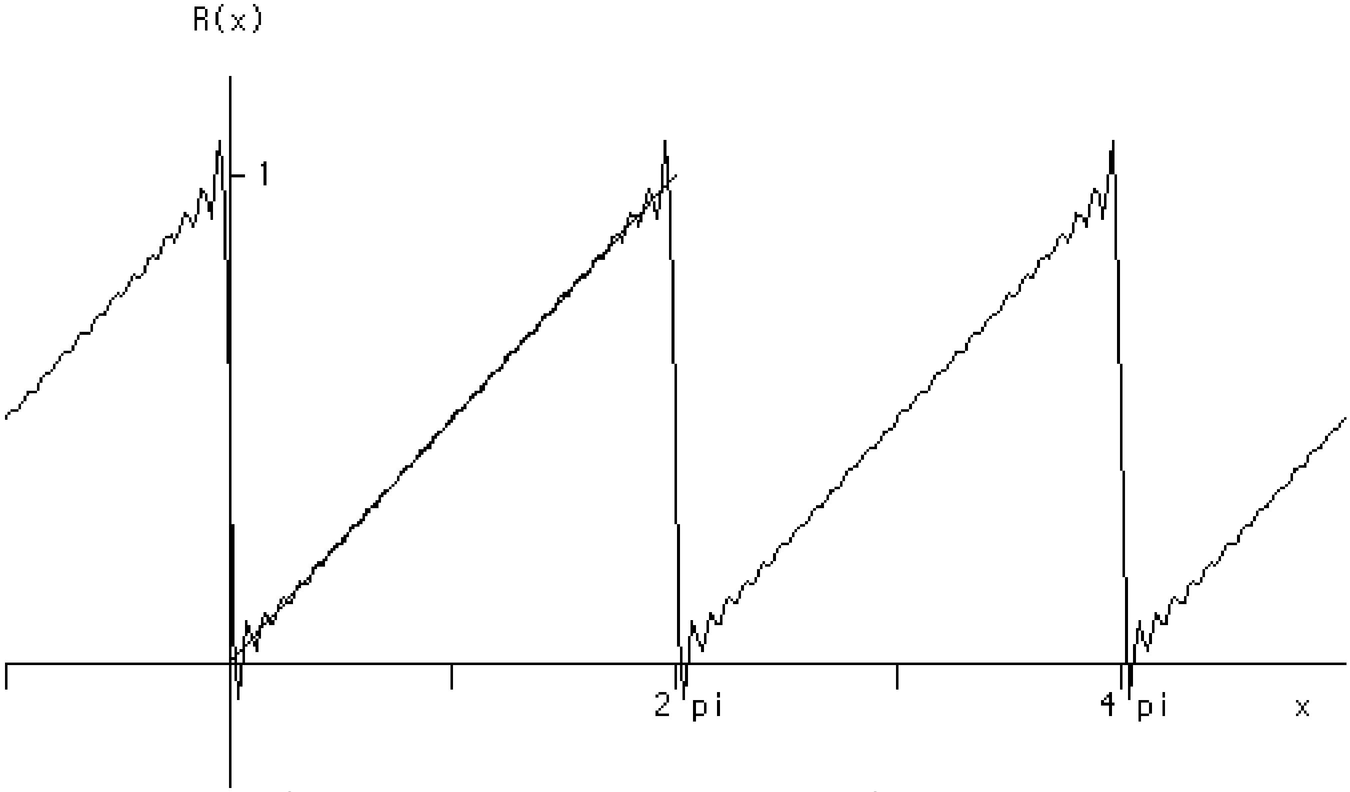

Fig. 3: Fourier sum for R(x) with n limited to a maximum of 25.

The next question is how many frequencies you would need to use to get a good ramp function. The answer is quite a lot. Consider fig. 3 which shows the same addition as in fig. 2, but including terms up to n=25. R(x) itself has also been plotted for the interval [0, 2π] only. Note that near the jump from 1 to 0 the Fourier sum displays significant oscillations away from the expected function. In fact, comparing them to those also visible in fig. 2 should convince you that the magnitude of the maximum error does not decrease much at all between n=6 and n=25, but the range over which the discrepancy occurs is decreased as n increases. These variations are known as *Gibbs' Oscillations* after the first person to discuss them, and you can see them in real life by hooking up a function generator to a good oscilloscope.

References: J. F. James, *A Student's Guide to Fourier Transforms*