Eigenvalues and Eigenvectors

Let's begin with an example. Suppose there is a cartesian coordinate system with a mirror placed along the line y = -x.

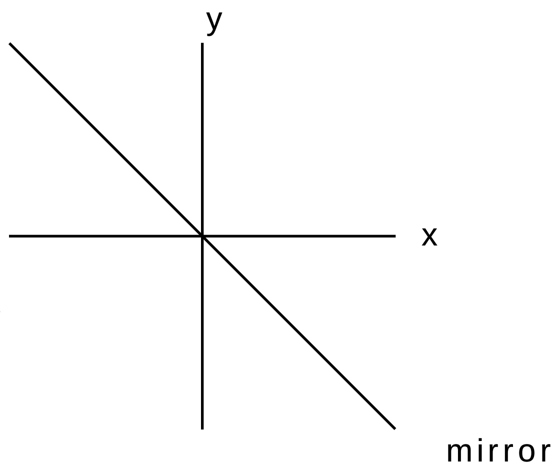


Figure 1

How are vectors above the mirror reflected into the bottom region? Take, for example, the vector (0, 1) shown. The angle that it makes with our mirror is preserved when it is reflected, so the reflection is the vector (-1, 0) also shown.

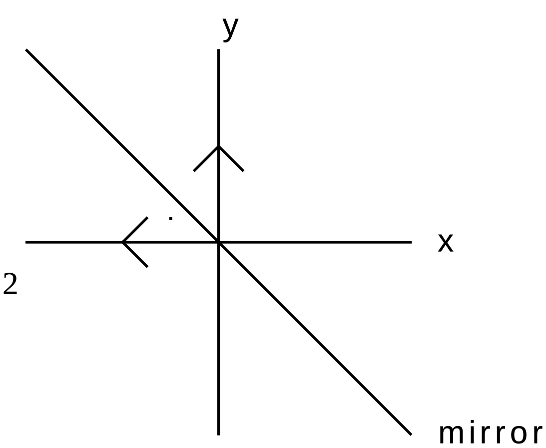


Figure 2

The mirror can be regarded as a transformation that takes vectors into the plane into other vectors. Any vectors that are either unchanged, or just reversed in direction are the characteristic vectors or eigenvectors of the transformation. The vectors shown in fig. 3 are obviously the eigenvectors associated with the mirror transformation.

Let's look at this in more mathematical detail. Let $\mathbf{r} \equiv (x, y)$ denote a vector in the plane. Then the mirror transformation can be implemented by a matrix **P** such that

$$\mathbf{r}' = \mathbf{P}\,\mathbf{r}$$

is the reflected vector. You can verify by checking some cases that for the mirror of fig. 1, **P** is given by

$$\mathbf{P} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

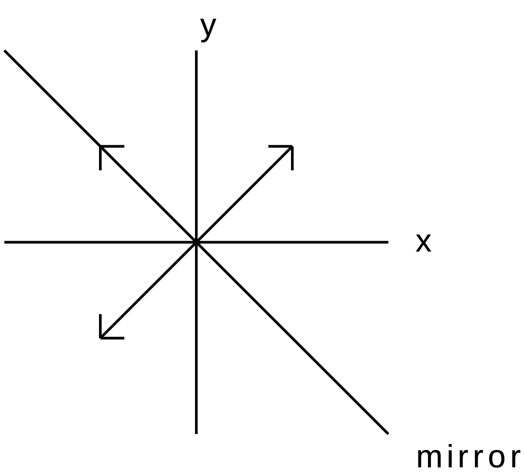
The reflection depicted in fig. 2 can be written

$$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

The eigenvectors, **r**, clearly correspond mathematically to the solutions of the equation

$$\mathbf{P}\,\mathbf{r} = p\,\mathbf{r} \qquad (1)$$

where p is a scalar, known as the eigenvalue as associated with eigenvector **r**. In general the eigenvalues can take on any value, leaving **P r** either parallel or antiparallel to **r**, but in this particular case the possible values are ±1.



Figure

Eq. (1) implies that

$$(\mathbf{P} - p\,\mathbf{I})\,\mathbf{r} = 0 \qquad (2)$$

where **I** is the identity matrix. For eq.(2) to hold, at least if $\mathbf{r} \neq 0$, it must be the case that

$$\det(\mathbf{P} - p\mathbf{I}) = 0 \qquad (3)$$

Eq. (3) allows us to determine all the eigenvalues p of the matrix **P**, and from them we can find the associated eigenvectors using eq. (1). Let's carry this out for the simple reflection matrix defined above.

$$\det(\mathbf{P} - p\mathbf{I}) = \det\begin{pmatrix} 0-p & -1 \\ -1 & 0-p \end{pmatrix} = p^2 - 1 = 0 \tag{4}$$

The solutions of eq.(4) are clearly p = ±1 and we next determine the eigenvector associated with each case. For p=1, using eq.(1) we have

$$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 1\begin{pmatrix} x \\ y \end{pmatrix}, \tag{5}$$

which implies that -y = x and -x = y, so that the eigenvector is $\mathbf{r}_1 = (-\alpha, \alpha)$.

For p = -1 a similar computation gives $\mathbf{r}_2 = (\alpha, \alpha)$. It is generally the case that the 'direction' of eigenvectors is determined by eq.(1), but not their magnitude, as one might also expect from the reflection example.

How does this relate to the eigenvalue problems which one encounters in quantum mechanics and classical mechanics? Eq. (1) is usually used in that form in classical mechanics and can be translated into the bracket notation of quantum mechanics as,

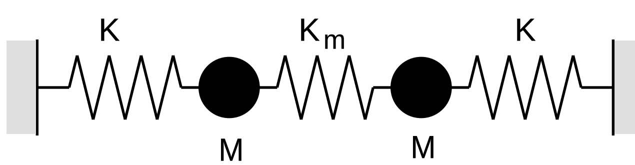
$$\mathbf{P}\,|r\rangle = p\,|r\rangle \tag{6}$$

where **P** is a matrix or an operator, $|r\rangle$ is a state vector in a Hilbert space, and p is the eigenvalue, a constant. Some examples will show the usefulness of the eigenvalue problem in these two areas.

Example 1

Consider a system of two coupled masses connected by a spring and each connected to a rigid wall. These look something like this:

Figure 4



Let $x_1(t)$ and $x_2(t)$ denote the time dependent displacements of the two masses away from their equilibrium positions. The equations of motion for x_1 and x_2 follow from Hooke's Law (the force exerted by a spring is proportional to its compression or extension) and Newton's 2nd law.

$$M\ddot{x}_1 = -K x_1 + K_m (x_2 - x_1) \tag{7a}$$

$$M\ddot{x}_2 = -K x_2 + K_m (x_1 - x_2) \tag{7b}$$

where M is the mass of each oscillator, K is the spring constant of the springs attached to the wall, and K_m is the spring constant of the middle spring. If we identify x_1 and x_2 as the components of a 2-dimensional vector, $\mathbf{x}(t) \equiv \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$, the eqs.(7) can be replaced by a single matrix equation.

$$- M \frac{d^2\mathbf{x}}{dt^2} = \begin{pmatrix} K + K_m & - K_m \\ - K_m & K + K_m \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{8}$$

To solve this equation we look for simple oscillatory solutions of the form

$$\mathbf{x}(t) = \mathbf{x}_o \exp(i\omega t), \tag{9}$$

where $\mathbf{x}_o$ gives the initial values of x_1 and x_2. (Do not be concerned about the use of a complex oscillatory solution here. One has in mind taking the real part at the end of the problem; the complex form simplifies the handling of phase factors.) Substituting eq.(9) into eq.(8) gives

$$M \omega^2 \mathbf{x}_o \exp(i\omega t) = \begin{pmatrix} K + K_m & - K_m \\ - K_m & K + K_m \end{pmatrix} \begin{pmatrix} x_{o1} \\ x_{o2} \end{pmatrix} \exp(i\omega t),$$

or

$$\begin{pmatrix} K + K_m - M\omega^2 & - K_m \\ - K_m & K + K_m - M\omega^2 \end{pmatrix} \begin{pmatrix} x_{o1} \\ x_{o2} \end{pmatrix} = 0 \tag{10}$$

after the exponential factor has been dropped from both sides. Eq. (10) is an eigenvalue problem with $M\omega^2$ playing the role of p in eq.(1) or (5) and we can proceed to solve it as we did those situations.

There will be two eigenvalues ω_1 and ω_2, each with a corresponding eigenvector. These eigenvalues determine the characteristic frequencies of the two possible fundamental motions that the oscillators can have. They can move back and forth together or they could be out of phase (as one moves to the right, the other goes to the left). Any other possible motion is a superposition of these two. These two motions are called the normal modes of the system.

The coefficients of equations of motions form the matrix **P** in eq. (1) and it is the determinant of that matrix which must be equal to zero.
Solving for ω, we find that

$$\omega = \sqrt{\frac{K + K_m \pm K_m}{M}} \, . \tag{11}$$

The two eigenfrequencies are, therefore,

$$\omega_1 = \sqrt{\frac{K}{M}} \quad \text{and} \quad \omega_2 = \sqrt{\frac{K + 2K_m}{M}} \, . \tag{12}$$

The corresponding eigenvectors turn out to be $\begin{pmatrix}1\\1\end{pmatrix}$ and $\begin{pmatrix}1\\-1\end{pmatrix}$ respectively. The components of the eigenvectors indicate the relative displacements of the two masses for each normal mode.

Example 2

Schrödinger's equation, encountered in quantum mechanics provides another example of an eigenvalue problem. The one-dimensional, time-independent case is usually written

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V(x)\,\psi(x) = E\,\psi(x) \tag{13}$$

where $\psi(x)$ is the wavefunction, $V(x)$ is the potential energy and E is the energy. To see that eq.(13) represents an eigenvalue problem like that of eq.(1), consider the expansion of $\psi(x)$ in terms of an orthonormal basis of functions, $\psi_n(x)$ that obey the boundary conditions associated with $V(x)$. Then any solution of eq.(13) can be written as a linear combination of the ψ_n's

$$\psi(x) = \sum_n a_n\,\psi_n(x)\,. \tag{14}$$

The expression for ψ can be substituted into eq.(13)

$$\hat{\mathbf{H}} \sum_n a_n\,\psi_n(x) = E \sum_n a_n\,\psi_n(x)\;, \tag{15}$$

where $\hat{\mathbf{H}}$ denotes the Hamiltonian operator

$$\hat{\mathbf{H}} \equiv -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x)\,. \tag{16}$$

$\hat{\mathbf{H}}$ can be taken inside the sum in eq.(15) to act on the individual ψ_n's so that

$$\sum_n a_n\,\hat{\mathbf{H}}\,\psi_n(x) = E \sum_n a_n\,\psi_n(x)\,. \tag{17}$$

Now we multiply on the left by $\psi_m^*(x)$ and integrate over all x.

$$\sum_n a_n H_{mn} = E \sum_n a_n \int \psi_m^*(x)\, \psi_n(x)\, dx \,, \tag{18}$$

where H_{mn} are matrix elements of the Hamiltonian operator

$$H_{mn} = \int \psi_m^*(x)\, \hat{\mathbf{H}}\, \psi_n(x)\, dx \;. \tag{19}$$

The right hand side of eq.(18) may be evaluated by the orthonormality of the ψ_n's, so that eq.(18) becomes

$$\sum_n a_n H_{mn} = E \sum_n a_n \delta_{m,n} \;. \tag{20}$$

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Error!

$$\sum_n a_n H_{mn} = E\, a_m. \tag{21}$$

The set of coefficients a_n may be taken to constitute a column vector **a**

$$\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ . \\ . \\ . \end{pmatrix} , \tag{22}$$

and then eq.(20) is seen to be a normal matrix eigenvalue problem

$$\mathbf{H}\,\mathbf{a} = E\,\mathbf{a} \,, \quad (23$$

where **H** is the matrix whose elements are the H_{mn}'s of eq.(19). E is an eigenvalue, and each eigenvector $\mathbf{a_n}$ determines and eigenstate of the Hamiltonian operator. (Recovering the function form of the eigenstate requires the use of eq.(14).)

For more detailed coverage of eigenvalue problems, see Boas and Arfken. Classical mechanical applications can be found in Marion, for example, and for the quantum mechanical side Liboff has a good treatment.