Differentiation and the Chain Rule

The chain rule is a technique for changing variables when taking derivatives. Suppose that you have a particle moving in the x-y plane. Suppose that its position is given as a function of time by

$$x(t) = A\cos(\omega t) \text{ and} \qquad (1)$$

$$y(t) = A\sin(\omega t),$$

you might want to find the motion in the y-direction relative to the x-direction, instead of relative to time. You can calculate the speed in both directions by taking the time derivatives, but how do you relate them? By using the chain rule, you can find dy / dx:

$$dx = -\omega A \sin\omega t\, dt \ \text{ and } dy = \omega A \cos\omega t\, dt \qquad (2)$$

$$\frac{dy}{dx} = \left(\frac{dy}{dt}\right)\left(\frac{dt}{dx}\right) = -\cot\omega t \qquad (3)$$

The general form of the chain rule is df / dy = (df / dg) (dg / dy), where f is an explicit function of g, and g is an explicit function of y. f also depends on y, but you use the chain rule when you only know the dependence through the dependence of f on g and g on y.

To apply the chain rule to multivariable cases, you just do the same thing. For example, let's derive the expression for the Laplacian, ^2V in spherical coordinates. The transformation from cartesian to spherical coordinates is the following (see the section on Spherical and Cylindrical Coordinates) :

$$x = r\cos\varphi\sin\theta,$$

$$y = r\sin\varphi\sin\theta, \qquad (4)$$

$$z = r\cos\theta.$$

The expression for the Laplacian in cartesian coordinates is

$$^2V = \frac{\partial^2 V_x}{\partial x^2} + \frac{\partial^2 V_y}{\partial y^2} + \frac{\partial^2 V_z}{\partial z^2} \qquad (5)$$

To find the expression for the Laplacian in spherical, do the chain rule on the expression for the Laplacian in cartesian.

$$\begin{aligned}
\left(\frac{\partial}{\partial x}\right)^2 &= \left(\frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial x}\frac{\partial}{\partial \theta} + \frac{\partial \varphi}{\partial x}\frac{\partial}{\partial \varphi}\right)^2 \\
\left(\frac{\partial}{\partial y}\right)^2 &= \left(\frac{\partial r}{\partial y}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial y}\frac{\partial}{\partial \theta} + \frac{\partial \varphi}{\partial y}\frac{\partial}{\partial \varphi}\right)^2 \\
\left(\frac{\partial}{\partial z}\right)^2 &= \left(\frac{\partial r}{\partial z}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial z}\frac{\partial}{\partial \theta}\right)^2
\end{aligned} \tag{6}$$

There is no $\partial/\partial\varphi$ term in the expansion of $\partial/\partial z$ because z is not a function of φ ($\partial z/\partial\varphi = 0$). Expanding this explicitly for $(\partial/\partial x)^2$, we get

$$\begin{aligned}
\left(\frac{\partial}{\partial x}\right)^2 = {} & \cos^2\varphi \sin^2\theta \frac{\partial^2}{\partial r^2} + r^2 \cos^2\varphi \cos^2\theta \frac{\partial^2}{\partial \theta^2} + r^2 \sin^2\varphi \sin^2\theta \frac{\partial^2}{\partial \varphi^2} \\
& + 2r \cos^2\varphi \sin\theta \cos\theta \frac{\partial}{\partial r}\frac{\partial}{\partial \theta} - 2r \cos\varphi \sin\varphi \sin^2\theta \frac{\partial}{\partial r}\frac{\partial}{\partial \varphi} \\
& - 2r^2 \cos\varphi \sin\varphi \cos\theta \sin\theta \frac{\partial}{\partial \theta}\frac{\partial}{\partial \varphi}
\end{aligned}$$

After calculating this for the $\partial/\partial y$ and $\partial/\partial z$ terms and adding all three together, we end up with the expression for the Laplacian in spherical coordinates:

$$\left(\frac{\partial}{\partial x}\right)^2 + \left(\frac{\partial}{\partial y}\right)^2 + \left(\frac{\partial}{\partial z}\right)^2 = \left(\frac{\partial}{\partial r}\right)^2 + r^2\left(\frac{\partial}{\partial \theta}\right)^2 + r^2 \sin^2\theta \left(\frac{\partial}{\partial \varphi}\right)^2 \tag{7}$$

By applying the right-hand side to V expressed in spherical coordinates, one can find the Laplacian in spherical coordinates.