Intro To Differential Equations

Differential equations are a mysterious subject at first. Ones professors usually produce "solutions" from out of the air. So we thought we would explain what ordinary differential equations are, how and why they're used, and the beginnings of how to solve them. Much of this section consists of the solving of one example, an equation arising from an LRC circuit.

Differential equations are of interest because they allow us to include the rate of change of a variable in a function, along with the variable itself. An equation is a partial differential equation if it includes partial derivatives, otherwise it is called an ordinary differential equation. If the dependent variable and its derivatives are first-order, then the differential equation is linear. For example, if there are any terms involving I^2 or $(dI/dt)^2$ in the equation in the LRC circuit example below, it will be a non-linear equation. As it happens, there are only terms with I and dI/dt, making it a linear equation. Only ordinary linear differential equations (excepting a few of the cases with first-order equations) will be dealt with in this section, for they're by far the easiest to solve and commonly occur in physics problems.

In a simple LRC electrical circuit, we have an inductor L, a resistor R, a capacitor C, and a (possibly time-dependent) source of emf V(t). The voltage drop across the resistor is IR, that across the inductor is L(dI/dt), where I is the current in the circuit, and across the capacitor the voltage drop is Q/C, Q being the charge on the capacitor. By Kirchoff's Law, we know that the sum of the voltages over all the elements in a closed circuit is zero. So,

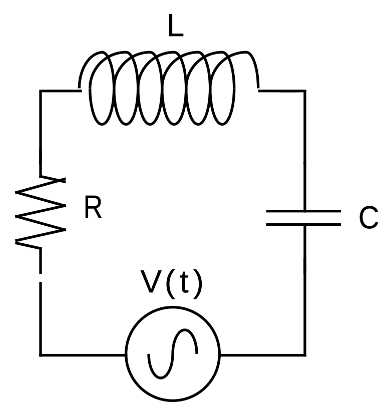

Figure 1

$$RI + L\frac{dI}{dt} + \frac{Q}{C} - V(t) = 0 . \qquad (1)$$

If we move V(t) to the right side, differentiate each term with respect to time and recall the definition of current I = dQ/dt, we obtain

$$L\frac{d^2 I}{dt^2} + R\frac{dI}{dt} + \frac{I}{C} = \frac{dV}{dt} . \qquad (2)$$

This is a second-order linear differential equation for the time-dependent current I(t). In order to solve this equation, we rewrite it in new derivative notation, letting **D** stand for d/dt:

$$(L\mathbf{D}^2 + R\mathbf{D} + \frac{1}{C})\, I = \frac{dV}{dt} . \qquad (3)$$

Now we put the eqn. into **normal** form by dividing by the leading coefficient L:

$$(\mathbf{D}^2 + \frac{R}{L}\mathbf{D} + \frac{1}{LC})\, I = \frac{1}{L}\frac{dV}{dt} . \qquad (4)$$

(Here, $L = (\mathbf{D}^2 + (R/L)\mathbf{D} + 1/LC)$ is a **linear differential operator**.)

To solve the equation, we must first put it into **homogeneous** form, meaning that the right-hand side is set equal to zero.

$$(\mathbf{D}^2 + \frac{R}{L}\mathbf{D} + \frac{1}{LC})\, I = 0 . \qquad (5)$$

We notice that the differential operator $\mathbf{D}^2 + (R/L)\mathbf{D} + 1/LC$ looks like a polynomial. We can take an analogous **characteristic polynomial** and factor it to find the roots of the differential eqn.

$$P(r) = r^2 + \frac{R}{L} r + \frac{1}{LC} = 0 . \qquad (6)$$

After factoring the characteristic polynomial, we have

$$r_1 = -\frac{R}{2L} + \frac{1}{2}\sqrt{\frac{R^2}{L^2} - \frac{4}{LC}} , \quad r_2 = -\frac{R}{2L} - \frac{1}{2}\sqrt{\frac{R^2}{L^2} - \frac{4}{LC}} . \qquad (7)$$

Now our factored homogeneous differential equation can be written:

$$(\mathbf{D} - r_1)(\mathbf{D} - r_2)\, I = 0. \qquad (8)$$

$(\mathbf{D} - r_1)\, I = 0$ or $(\mathbf{D} - r_2)\, I = 0.$

$\mathbf{D}\, I = r_1 I \qquad \mathbf{D}\, I = r_2 I$

$dI/dt = r_1 I$.

$dI = r_1 I\, dt$.

$$\int \frac{1}{I}\, dI = \int r_1\, dt \quad .$$

$\ln I = r_1 t + C_1$.

$$I = e^{r_1 t} + e^{C_1} = k_1 e^{r_1 t} \qquad I = k_2 e^{r_2 t} \qquad (9)$$

(k_1 and k_2 are complex constants.)

This explains why, in the following discussion, we get exponential solutions from the roots of a characteristic polynomial. If $R^2/L^2 - 4/LC < 0$, then the exponentials are complex. This implies oscillatory solutions

$$I_1(t) = k_1\, e^{\left(-\frac{R}{2L} + \frac{i}{2}\sqrt{\frac{4}{LC} - \frac{R^2}{L^2}}\right)t} \; ; \; I_2(t) = k_2\, e^{\left(-\frac{R}{2L} - \frac{i}{2}\sqrt{\frac{4}{LC} - \frac{R^2}{L^2}}\right)t} \; . \qquad (10)$$

We have two solutions for I(t), and their linear combination is the general solution of the homogeneous equation.

$$I(t) = k_1\, e^{\left(-\frac{R}{2L} + \frac{i}{2}\sqrt{\frac{4}{LC} - \frac{R^2}{L^2}}\right)t} + k_2\, e^{\left(-\frac{R}{2L} - \frac{i}{2}\sqrt{\frac{4}{LC} - \frac{R^2}{L^2}}\right)t} \; . \qquad (11)$$

Eq. (11) is complex and generally one expects currents to be real. This just means that one needs to be careful in choosing the constants k_1 and k_2, which have so far not been specified. Recall Euler's formula, which can be adapted to the present case as follows,

$$e^{(\alpha + \beta i)} = e^{\alpha}(\cos\beta + i\sin\beta). \qquad (12)$$

Defining α and β appropriately we rewrite eq. (11) as

$$I(t) = k_1 e^{(\alpha + i\beta)t} + k_2 e^{(\alpha - i\beta)t} \qquad (13a)$$

$$= e^{\alpha t}\,[(k_1 + k_2)\cos\beta t + i\,(k_1 - k_2)\sin\beta t] \; . \qquad (13b)$$

Thus, I(t) is real if $k_1=k_2$, or if $k_1 = - k_2 = i\kappa$ where κ is a real number. If one wishes to obtain a solution of some other phase, i.e. with time dependence cos (βt + φ), other choices of the k's will give the desired result.

Note: there must be as many general solutions for a linear differential equation as the order of the equation, two in our example.

Now that we have the solution to the homogeneous equation (Eq. 12), we return to the original equation

$$(\mathbf{D}^2 + \frac{R}{L}\,\mathbf{D} + \frac{1}{LC})\, I \;=\; \frac{1}{L}\frac{dV}{dt}\,. \tag{4}$$

In order to find a solution specific to this equation, we need a particular solution in addition to the general solution of the homogeneous equation which has been found. We look at the right side of Eq. 4 and figure out which differentiating operator could turn it into zero. This is the **annihilating** operator. At this point, we need to know V(t) as a specific function of t. To look at a particular case, let's take

R, L, C = 1 and V(t) = 10 sin 2t.

So that eq.(11) becomes

$$I(t) = k_1\, e^{\left(\frac{1}{2} + \frac{i}{2}\sqrt{3}\right)t} + k_2\, e^{\left(\frac{1}{2} - \frac{i}{2}\sqrt{3}\right)t}\,. \tag{14}$$

And

$$\frac{1}{L}\frac{dV}{dt} \;=\; 20 \cos 2t. \tag{15}$$

We find an annihilating operator ($\mathbf{D}^2$ + 4) for (15) and test it:

$$\begin{aligned}(\mathbf{D}^2 + 4)(20 \cos 2t) &= (\ \mathbf{D}(-40 \sin 2t) + 80 \cos 2t) \\ &= (-80 \cos 2t + 80 \cos 2t) \\ &= 0.\end{aligned}$$

The operator indeed annihilates 20 cos 2t.

Brief guidelines for finding annihilating operators:

(D) will annihilate a constant;

$(D \mp m)$, " $e^{\pm m t}$

$D^n (D \mp m)$, " $e^{mt} + t^{n-1}$

$(D^2 + m^2)$, " sin mt or cos mt
and so on.

We apply the annihilating operator to both sides of the normal equation:

$$\left(\left(\mathbf{D}^2 + 4\right)\left(\mathbf{D}^2 + \frac{R}{L}\,\mathbf{D} + \frac{1}{LC}\right)\right) I \;=\; (\mathbf{D}^2 + 4)\left(\frac{1}{L}\frac{dV}{dt}\right) \qquad (16)$$

Substituting in the values for R, L, C, and V,

$$(\mathbf{D}^2 + 4)(\mathbf{D}^2 + \mathbf{D} + 1)\, I \;=\; 0. \qquad (17)$$

We solve this equation in the same way as above and find that

$$r_1 = -\frac{1}{2} + \frac{i\sqrt{3}}{2}, \quad r_2 = -\frac{1}{2} - \frac{i\sqrt{3}}{2}, \quad r_3 = 2i, \quad r_4 = -2i .$$

So I(t) can be written

$$I(t) = k_1\, e^{\left(-\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)t} + k_2\, e^{\left(-\frac{1}{2} - \frac{i\sqrt{3}}{2}\right)t} + k_3\, e^{2it} + k_4\, e^{-2it}. \qquad (18)$$

The first two terms constitute the general solution again, so we focus for the moment on the new terms.

Particular solution:

$$I_p(t) \;=\; k_3\, e^{2it} + k_4\, e^{-2it}$$

$$= c_3 \cos 2t + c_4 \sin 2t \qquad (19)$$

using Euler's equation again.
We can find the coefficients c_3 and c_4 by applying the original differentiating operator *L* to this solution of I. In the normal equation, *L*I = (1/L) dV/dt, so

$$\left(\mathbf{D}^2 + \frac{R}{L}\,\mathbf{D} + \frac{1}{LC}\right) I \;=\; \frac{1}{L}\frac{dV}{dt} \qquad (4)$$

$$(\mathbf{D}^2 + \mathbf{D} + 1)\,(c_3 \cos 2t + c_4 \sin 2t) \;=\; 20 \cos 2t \qquad (20)$$

$$(-4\, c_3 \cos 2t - 4\, c_4 \sin 2t) + (-2\, c_3 \sin 2t + 2\, c_4 \cos 2t)$$

$$+ (c_3 \cos 2t + c_4 \sin 2t) \;=\; 20 \cos 2t$$

We should then require

$$-3\,c_3 + 2\,c_4 = 20; \qquad -2\,c_3 - 3\,c_4 = 0\,. \tag{21}$$

Solving these equations simultaneously, we get

$$c_3 = -60/13 \quad \text{and } c_4 = 40/13.$$

So our particular solution is

$$I_p(t) = -\frac{60}{13}\cos 2t + \frac{40}{13}\sin 2t. \tag{22}$$

Our complete solution is the superposition of the general and particular solutions:

$$I(t) = k_1\, e^{\left(\frac{1}{2}+\frac{i}{2}\sqrt{3}\right)t} + k_2\, e^{\left(\frac{1}{2}-\frac{i}{2}\sqrt{3}\right)t} - \frac{60}{13}\cos 2t + \frac{40}{13}\sin 2t. \tag{23}$$

If we know the current I and its rate of change $\frac{dI}{dt}$ at some time, we can also solve for the two remaining constants k_1 and k_2.

Second-Order Equations:

We start with methods for solving second-order equations, because they turn up a lot in physics. Here we separate methods according to whether they can be used on homogeneous or non-homogeneous equations.

Homogeneous:

A homogeneous equation with constant coefficients can be solved as shown above. Briefly reiterating, we divide by the leading coefficient, put the equation into **D** notation, and use the characteristic polynomial to find the roots r_1 and r_2. From that we get the exponential solutions $x = k_1e^{r_1t} + k_2e^{r_2t}$. If the roots are complex, we express the solutions in terms of sines and cosines. If r_1 and r_2 are the same, then the second term in the general solution has an extra factor of t.

Example:

$(\mathbf{D}^2 - 2\mathbf{D} + 1)x = 0.$

$r^2 - 2r + 1 = 0.$

$r_1 = 1 \quad r_2 = 1$

$x = k_1e^t + k_2te^t.$

When the original differential equation is homogeneous, then the general solution is the complete solution. Note that the coefficients of the differential equation must be constants to use this method.

Non-homogeneous:

If your differential equation is non-homogeneous, i.e., the right side is not zero, there are two popular methods by which you may solve your differential equation. The method of **undetermined coefficients** requires constant coefficients, and a certain form for the right side (namely, that it can be annihilated). Undetermined coefficients was demonstrated in the LRC circuit example. The method of **variation of parameters** effectively requires constant coefficients, for you need to know the general solution of the homogeneous eqn, and here we have only shown how to find that for an equation with constant coefficients. But if you already know the general solution of the homogeneous equation, then there is no barrier to your using variation of parameters for an equation with variable coefficients.

Variation of Parameters:

This does not require E(t) to be in a form which can be annihilated, so it's a better (though longer) method to use with a complicated expression on the right side.

Go through the usual procedure: put equation into **D** notation, make it normal, then make it homogeneous, and solve the homogeneous equation. This should give you two roots, so that the general solution is:

$$x = H(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} = c_1 h_1(t) + c_2 h_2(t). \qquad (24)$$

Since we didn't stipulate that the equation's coefficients be constant,possibly the c's are not constants either. In this case, we want to find a particular solution of the normal equation:

$$x = p(t) = c_3(t)h_1(t) + c_4(t)h_2(t). \qquad (25)$$

To find $c_3(t)$ and $c_4(t)$, we set up a system of 2 equations {much math is required to show why we do this}:

$$c_3'(t)h_1(t) + c_4'(t)h_2(t) = 0. \qquad (26a)$$

$$c_3'(t)h_1'(t) + c_4'(t)h_2'(t) = q(t). \qquad (26b)$$

We can find $c_3'(t)$ and $c_4'(t)$ by Cramer's Rule {you can look this up in the references}

$$c_3'(t) = \frac{\det\begin{bmatrix} 0 & h_2(t) \\ q(t) & h_2'(t) \end{bmatrix}}{\det\begin{bmatrix} h_1(t) & h_2(t) \\ h_1'(t) & h_2'(t) \end{bmatrix}} \qquad (27a)$$

$$c_4'(t) = \frac{\det\begin{bmatrix} h_1(t) & 0 \\ h_1'(t) & q(t) \end{bmatrix}}{\det\begin{bmatrix} h_1(t) & h_2(t) \\ h_1'(t) & h_2'(t) \end{bmatrix}} \qquad (27b)$$

Integrate to find $c_3(t)$ & $c_4(t)$. Then the general solution of the normal equation is:

$$x = H(t) + p(t) = c_1 h_1(t) + c_2 h_2(t) + c_3(t) h_1(t) + c_4(t) h_2(t). \qquad (28)$$

Again, we will know $c_3(t)$ and $c_4(t)$, the coefficients of the particular solution, but not c_1 or c_2.

First-Order Equations

Now, we look at first-order differential equations, which commonly turn up as models for population growth, radioactive decay, etc. We discuss how to solve them by separation of variables, which can't be applied in all cases, and variation of parameters, which can, for all linear differential equations.

Separation Of Variables:

One of the simplest ways to solve certain differential equations is separation of variables, which can be used only with first-order differentiall equations (not necessarily linear). For this technique, the equation must be able to be written in the form

$$dx / dt = f(x)\, g(t), \qquad (29)$$

where x and t are the generic dependent and independent variables, respectively. For example, if you have the equation:

$$x^2\, (dx / dt) = t^3 + t\ ,$$

make this into : $dx / dt = (t^3 + t)\,(1 / x^2)$ and we see that the form is correct.

Convert it into : $x^2\, dx = (t^3 + t)\, dt$.

Integrate both sides: $\int x^2\, dx = \int (t^3 + t)\, dt$

$(1/3)x^3 + C = (1/4)t^4 + (1/2)t^2 + B$,

where B & C are arbitrary constants of integration.

Combine the constants of integration, and let B - C = G.

$$(1/3)x^3 = (1/4)t^4 + (1/2)t^2 + G .$$

$$x = ((3/4)t^4 + (3/2)t^2 + G'))^{(1/3)} , \quad \text{where } G' = 3G.$$

Supposing we had had the initial conditions $x(1) = 2$, we would be able to solve for G'.

$2 = ((3/4)(1) + (3/2)(1) + G')^{(1/3)}$
$8 = 3/4 + 3/2 + G'$
$G' = 5.75$

$$x = ((3/4)t^4 + (3/2)t^2 + 5.75)^{(1/3)} \qquad (30)$$

Often, a non-separable equation can be put into separable form by substitutions–an example of this is the method of Homogeneous Coefficients.

Homogeneous Coefficients:

If the first-order differential equation can be written as

$dx/dt = g(x/t)$,

where <u>every</u> term is a function of x/t , then let

$v = x/t$,
$x = vt$, and
$dx/dt = d/dt(vt) = v + t(dv/dt)$.

Example:

$x^2\, t\, (dx/dt) + t^3 - x^3 = 0$
$x^2\, t\, (dx/dt) = -t^3 + x^3$
$(dx/dt) = -(t^2/x^2) + (x/t)$
$= -v^2 + v$

$$v + t(dv/dt) = -v^2 + v$$
$$t(dv/dt) = -v^2$$

This is now in separable form, and we can integrate both sides.

$$- v^2 \, dv = (1/t)\, dt$$
$$-(1/3)v^3 = \ln t + C$$
$$-(1/3)(x/t)^3 = \ln t + C$$
$$x^3 = (-3 \ln t - 3C)(t^3)$$
$$x = t(-3 \ln t + C')^{(1/3)}$$

Higher Order Equations

Occasionally, higher-order differential equations can be solved by separation of variables, if substitutions such as y = dx/dt can be used to reduce them to first-order, or if a trick can be used, as below.

Example:

$$(d^2x/dt^2) = -1/(t+1)^2 \quad ; x(0) = 2,\ x'(0) = 3 .$$
$$(d/dt)(dx/dt) = -1/(t+1)^2$$
$$\int \frac{d}{dt}\frac{dx}{dt}\, dt = \int \frac{-1}{(t+1)^2}\, dt$$
$$\frac{dx}{dt} = \int \frac{-1}{(t+1)^2}$$

let u = t + 1
let du = 1 dt

$$\frac{dx}{dt} = \int \frac{-1}{u^2}\, du$$

$$dx/dt = 1/u + C.$$

This can now be solved by separation of variables.

$$\int dx = \int \left(\frac{1}{u} + C\right) dt$$
$$x = \ln u + Ct + C'$$

Putting in initial values tells us that C = C’ = 2.

$$x = \ln\ t + 1\ \ + 2t + 2\ .$$

References: Guterman & Nitecki, Differential Equations
See also: Simmons, Differential Equations with Applications & Historical Notes