Determinants

$$\det\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = aei - afh - bdi + bfg + cdh - ceg \tag{1}$$

But, you ask, what does this mean? This is called the determinant of the matrix. To show how to calculate it, let's begin with a 2 x 2 matrix.

$$\det\begin{bmatrix} D_1 & D_2 \\ E_1 & E_2 \end{bmatrix} \tag{2}$$

The determinant of this matrix is just the difference of the diagonal products, with the top left to bottom right diagonal being the positive term.

$$\det\begin{bmatrix} D1 & D2 \\ E1 & E2 \end{bmatrix} = D1E2 - D2E1 \tag{3}$$

Easy enough!

To handle larger matrices we use the technique called the expansion by minor determinants on cofactors. Consider our first example. To calculate this determinant, begin with the element in the upper left-hand corner. Ignore the elements in the same row and column as this element -- the ones in italics -- and multiply the 2x2 determinant in the lower right-hand column by the letter in boldface (here, a).

$$\begin{bmatrix} \mathbf{a} & \mathit{b} & \mathit{c} \\ \mathit{d} & \mathrm{e} & \mathrm{f} \\ \mathit{g} & \mathrm{h} & \mathrm{i} \end{bmatrix} \tag{4}$$

The first term, then, is a (ei - fh). Repeat for the other elements of the first row, but alternate signs.

$$\begin{bmatrix} a & \mathbf{b} & c \\ \mathrm{d} & e & \mathrm{f} \\ \mathrm{g} & h & \mathrm{i} \end{bmatrix} \quad \text{and} \begin{bmatrix} a & b & \mathbf{c} \\ \mathrm{d} & \mathrm{e} & f \\ \mathrm{g} & \mathrm{h} & i \end{bmatrix} \tag{5}$$

Finally, add all the terms to get the determinant. The determinant, then, is

$$\det = a(ei - fh) - b(di - fg) + c(dh - eg) \tag{6}$$

And this does, in fact, lead to the stated result! You can, in fact, use any row or column, not just the first row, as the one multiplying the minor determinants. In our example, we could also have used the d-e-f row, the g-h-i row, the a-d-g column, the b-e-h column, or the c-f-i column.

To find the determinant of a 3 x 3 matrix, you can also take the difference between the left-to-right diagonals and the right-to-left diagonals. Pictorially, this can be thought of as the following:

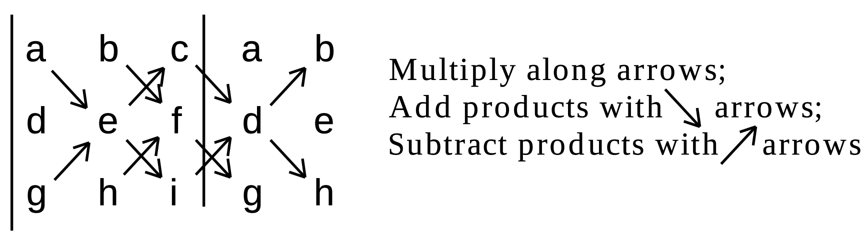

Fig. 1

Beware of trying to use the difference of the diagonals for larger matrices. While this works for a 3 x 3 matrix, it DOES NOT for any matrix larger than 3 x 3 !!! Let's see that by looking at a 4 x 4 matrix.

$$\det \begin{bmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{bmatrix} \tag{7}$$

By adding all the possible diagonals, we get the following:

$$\det = afkp + bglm + chin + dejo - dgjm - ahkn - belo - cfip$$

Now by doing the minor determinants method, we get:

$$\begin{aligned}\det = a[\, f \{ kp - lo \} - g \{ jp - ln \} + h \{ jo - kn \}] \\ - b [\, e \{ kp - lo \} - g \{ ip - lm \} + h \{ io - km \}] \\ + c [\, e \{ jp - ln \} - f \{ ip - lm \} + h \{ in - jm \}] \\ - d [\, e \{ jo - kn \} - f \{ io - km \} + g \{ in - jm \}]\end{aligned}$$

Whew! There are 24 elements here, as opposed to the eight in the diagonals method. The diagonals method only leads to 1/3 of the elements of the determinant, and so it's wrong!

Also also don't try to take a determinant of a matrix which is not square, i.e. a matrix with 3 rows and 2 columns. It won't work.

Calculating a determinant is also an easy way to remember how to take the cross product of two vectors.

$$\vec{A} \times \vec{B} = \vec{C} \tag{8}$$

How do you calculate this vector? Assuming that the basis vectors that you are working with are the Cartesian unit vectors **i**, **j**, and **k**, the solution is

$$\vec{C} = \det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{bmatrix} \tag{9}$$

$$= \mathbf{i}\,(A_yB_z - A_zB_y) + \mathbf{j}(A_zB_x - A_xB_y) + \mathbf{k}\,(A_xB_y - A_yB_x)$$

For a matrix larger than 3 x 3 or 4 x 4, I would suggest trying to calculate it by a different method, namely row reduction, instead of the method described above.

For more on determinants and cross products, see any linear algebra text or any advanced calculus text.