Notes on Binomial Expansions and Approximations*

You should know that there is a general way to calculate out the coefficients of the following expression:

$$(a + b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2!} a^{n-2}b^2 + ... \qquad (1)$$

This relation is the so-called binomial expansion. It certainly is an improvement over multiplying out (a+b)(a+b)...(a+b) by hand! The series in eq. (1) can be used for any value of n, integer or not, but when n is an integer the series terminates or ends after n+1 terms. For example

$$(a+b)^5 = a^5 + 5a^4b^1 + \frac{5.4}{1.2} a^3b^2 + \frac{5.4.3}{3.2.1} a^2b^3 + \frac{5.4.3.2}{4.3.2.1} a^1b^4 + \frac{5.4.3.2.1}{5.4.3.2.1} a^5$$

and the next term would vanish by extension.

$$= a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$$

Note that the coefficients of that occur in the binomial expansion formula (n+1)st row of Pascals triangle.

whose entries are obtained by adding the two above each one.

```
                  1
               1     1
            1     2     1
         1     3     3     1
      1     4     6     4     1
---->  1     5    10    10     5     1   <---- n = 5
   1     6    15    20    15     6     1
```

An even neater form of eq.(1) occurs when a = 1. Changing b to x:

$$(l + x)^n = 1 + nx + \frac{n(n-1)}{2!} x^2 + \frac{n(n-l)(n-2)}{3!} x^3 + ... \qquad (2)$$

We will find it very helpful to use this expansion under circumstances when x is small compared to one. An expansion like that of eq.(2) occurs, for example, based on eq.(1) if a is much greater than b. You can see this as

* Adapted from notes written by Charles Holbrow and J. N. Lloyd, in use at Colgate University

follows. Let $x \equiv b/a$ with $a >> b$ so that $x << 1$. Then the right hand side of eq.(1) can be rearranged:

$$(a+b)^n = a^n \left(1 + \frac{b}{a}\right)^n = a^n (1 + x)^n$$

When x is small, each successive term in the series, eq. (2), gets increasingly smaller by roughly a factor of x. Thus for moderate sizes of n:

$$nx << 1; \qquad \frac{n(n-1)}{2!} x^2 << nx; \qquad \text{and so on....}$$

At any point, if we just quit adding terms, the ones we have left out would have added hardly anything to the sum, so we don't make much of an error. Thus we can often make a good approximation by leaving out all but the first two or three terms.

Example 1

Let $n = 2$ and $x = .0200$
Exact: $(1 + .02)^2 = 1.0404$

Two term approx.: $1 + 2*.0200 = 1.0400$

Error: only 0.04%

Now comes something very interesting. Those people skilled at this sort of thing can tell us that the expression (2) is correct even when n is **not** a positive integer provided $x < 1$. The real power of the method now becomes evident.

Example 2

Let $n = -1$ and $x = 0.020$

$(1 + 0.02)^{-1} = 1/1.02$

Plugging $n = -1$ into (2) gives

$$1 + (-1)\,x + \frac{(-1)(-2)}{2} x^2 + \ldots$$

While the method now leads to an infinite series, we are assured by mathematicians that the series does accurately reproduce the function $\frac{1}{1+x}$ if (and only if) $|x| < 1$.

Again ignoring higher terms, we get:

$$(1.02)^{-1} = 1 + (-1)0.20 = 0.980$$

Accurate value: 0.98039, or again only a .04% error

Example 3

One final example occurs frequently in physics

$$\frac{1}{\sqrt{1 \pm x}} = (1 \pm x)^{-1/2} \approx 1 + (-\tfrac{1}{2})(\pm x) = 1 \mp \frac{x}{2} \quad (3)$$

Problem 1

Choose the + sign case and try the exact calculation for three assumed values of x and compare with the approximation. (See the figure on the following page for further interesting comparisons.)

Problem 2

This technique allows you to outwit your calculator in some extreme situations. Try:

$$1 - \frac{1}{\sqrt{1 - 1.5x10^{-12}}} = ? \qquad \text{(The answer is \underline{not} exactly zero!)}$$

The technique is also valuable for seeing how algebraic expressions behave without having to use specific values of x. Consider the form in Example 3. The dependence of the reciprocal square root on the quantity x would be very hard to visualize. But, as shown in the figure below for the function $\frac{1}{\sqrt{1+x}}$, when x is small, the approximation clearly shows the dependence on x to be linear and to have slope $-\frac{1}{2}$.

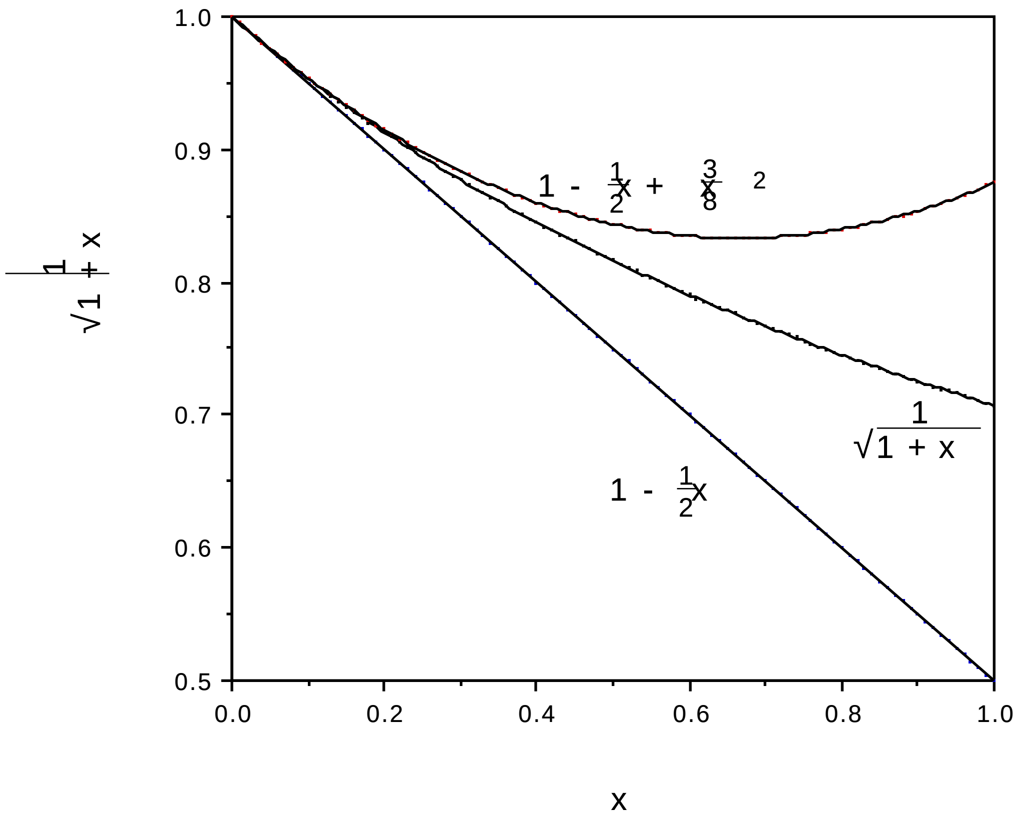

Figure: Plot of the function $\frac{1}{\sqrt{1+x}}$ and the 2nd and 3rd order approximations to it for small x.