applied shear rate is too small (such that $X << 1$ in Appendix A.1) compared to the minimum that is required for any individual grain to be temporarily independent of its neighbors.

As a result, the underlying physics of the dynamics of sheared quasi-static packing studied here is perhaps mainly about the *adjustment* of a packed structure driven under boundary constraints, as opposed to reflecting instantaneous competitions between different factors. The quasi-static adjustments are rate-independent and often irreversible. In Section 5.2.1, we illustrate the rate-independent adjustment of internal grains to a new direction of flow, in the form of an anomalous mobility during the shear-reversal transient. A fully non-linear model (that does not require a finite linear regime) may be necessary for describing this behavior. Interestingly, densely packed grains may adjust to alternating shearing with a fixed amplitude, as well as to a stationary shearing in a fixed direction - see the experiments reported in Ref. [42] as an example, where grains show stepwise descents in heights only when the amplitude of imposed cyclic shearing is abruptly change.

5.3.5 Granular flows vs. Fluid flows

How does the velocity field of a slowly sheared granular packing compare to a viscous fluid driven in the same geometry?

In Appendix A.2 we discuss the consequence of the fact that, in the simple case of an incompressible Newtonian fluid , the velocity field for the steady flow satisfies a 2D Laplace equation $\nabla^2 v_x = 0$ across a rectangular channel. This equation couples the second derivatives (the "curvatures") in two orthogonal directions. The vertical decay lengths are well coupled to the horizontal wave numbers that that are selected by the finite-width channel due to the no-slip boundary conditions. The principal decay length is therefore $1/\lambda_1 = (W_0/n\pi)|_{n=1} = W_0/\pi$. (See the slope on the semi-log plot Fig. 5.5). Note that this is exact only when the fluid is Newtonian. Otherwise, the more general stress balance (as in Eqn. A.7) requires

$$(\nabla\eta) \cdot (\nabla v_x) + \eta\nabla^2 v_x = 0 \tag{5.1}$$

on the yz plane rather than $\nabla^2 v_x = 0$. Here, one can see that the two "curvatures" of the velocity field are comparable ($\frac{\partial^2 v_x}{\partial z^2} \approx -\frac{\partial^2 v_x}{\partial y^2}$) when the non-Newtonian term of the equation can be ignored - this is true only when the characteristic length scale of variation in η is much longer than the cross-sectional dimension of the channel. Therefore the characteristic vertical decay length may still be comparable to the channel width if the fluid is only weakly non-Newtonian.

On the other hand, granular particles slip against the smooth sidewalls instead of satisfying no-slip boundary conditions for ordinary fluids. This observation may suggest that granular flow would not be slowed down by the sidewalls as much as ordinary fluids driven in the same channel. However, by comparing measured velocity fields of granular particles to the curves representing the ordinary-fluid flows (Fig. 5.5), we find that the downward decay of average velocity at each height for granular particles is indeed much steeper than that of ordinary fluid. This apparent conflict is resolved if one recognizes that granular flow is highly non-Newtonian so that the vertical (z) and horizontal (y) variations of velocity field do not need to be comparable. A quasi-statically sheared granular packing is obviously non-Newtonian, since the shear stress only has a very weak dependence on shear rate, if not vanishing. This fact is confirmed by our shear force measurements with different driving speeds, and by numerous previous experiments.

In the following discussions, I attempt a heuristic model showing that, if the internal shear stress of a material has only a weak dependence on the local shear rate, shear banding can occur under the influence of the frictional resistance imposed by the sidewalls. The simple model is based on coarse-grained assumptions of local rheology, and does not explicitly involve the size of particles.

Here we assume that the shear stress inside the material is related to the *local* shear rate. And we restrict the discussion to the cases where the shear stress monotonically increases with shear rate; this restriction ensures the stability of a stationary velocity profile inside the bulk.

We use the crystallized state as an example, for its simple feature that the velocity is uniform at each height. The shear stress $\sigma_{xz}(z)$ and the steady-state velocity $V_x(z)$ are therefore functions of height z only. The shear rate $\dot{\gamma} = V_x'(z)$ is simply the derivative of $V_x(z)$. The z axis points vertically upwards with $z = 0$ representing the driving boundary at the top. The shear resistance exerted by the sidewall per unit area is represented by

$\sigma^{Wall}(z)$. By considering the stress balance for a stationary state, we have

$$W_0 \cdot (\sigma_{xz}(z) - \sigma_0) = 2 \int_0^z \sigma^{Wall}(z')dz' \tag{5.2}$$

in which $\sigma_0 \equiv \sigma_{xz}(0)$. In [1] and [2] of the following discussions, we ignore the finite height H_0 by assuming $H_0 \to \infty$ and accepting the fact that V_x decreases as z goes away from the driving surface $z = 0$.

[1] To the first approximation, we assume that the resistance at the sidewalls is a constant

$$\sigma^{Wall}(z) = \sigma_0^W \tag{5.3}$$

so that the right-hand side of Eqn. (5.2) reduces to $2\sigma_0^W z$. In general, when one assumes a local relation $\sigma_{xz} = f(\dot{\gamma})$ where the value of $f(\dot{\gamma})$ slowly increases with increasing $\dot{\gamma}$, the solution of Eqn (5.2) leads to a shear-banding velocity profile with its gradient highly localized near $z = 0$.

For instance, $f(\dot{\gamma})$ can be a logarithmic function of shear rate $\dot{\gamma}$

$$f(\dot{\gamma}) \approx \sigma_0(1 + \alpha \ln(\dot{\gamma}/\dot{\gamma}_0)) \tag{5.4}$$

in which $\gamma_0 \equiv V_x'(0)$. The positive constant α is assumed to be much smaller than unity so that $1 + \alpha \ln(\dot{\gamma}/\dot{\gamma}_0)$ remain positive for the entire interested range of shear rate. (The notion of logarithmic dependence has been put forth by previous researchers; see Ref. [20].) Under these assumptions, one would obtain

$$V_x(z) = V_x(0) \cdot \exp\left(\frac{2\sigma_0^W}{\alpha\sigma_0 W_0} z\right) \tag{5.5}$$

which exhibits an exponential shear banding. Note that the decay length of this velocity profile is proportional to W_0, but is scaled down by a factor of α.

[2] An improvement of the model is to introduce a spatial variation of the resistance force at the sidewalls. In general, if $\sigma^{Wall}(z)$ gradually increases as z goes downwards, we can expect an increasingly negative slope on the semi-log plot of velocity profiles such as Fig. 5.4 and other figures. One possibility is to assume a "velocity weakening" frictional law $\sigma^{Wall} = g(v^{slip})$ with a logarithmic dependence on the slip velocity v^{slip}

$$g(v^{slip}) \approx \sigma_0^W (1 + \beta \ln (v^{slip}/{v^{slip}}_0)) \tag{5.6}$$

in which a small negative β characterize the slight decrease of force with respect to the slip velocity. For our crystallized state, we can assume $v^{slip}(z) = V_x(z)$ and make the problem self-content. Since there is no analytical solution to the resultant differential-integral equation [Eqn (A.9) in Appendix A.3], an approximate solution

$$\ln(V_x(z)/V_x(0)) \sim \frac{2\sigma_0^W}{\alpha\sigma_0 W_0} \cdot (z + \frac{\beta}{2}(\frac{2\sigma_0^W}{\alpha\sigma_0 W_0})z^2) \quad (5.7)$$

is derived using the iteration scheme described in Appendix A.3. Note that the quadratic form of this approximate solution qualitatively captures the shape of the master curve shown in Fig. 5.4. [3]

[3] Finally, to take into account effect of the rough bottom boundary at the finite depth $z = -H_0$, one would need to introduce an appropriate boundary condition that "penalizes" slip against a rough surface, i.e. the resistance increases with larger slip velocity. I expect that an additional 'velocity-strengthening' boundary condition [4] for a rough bottom should lead to a steeper velocity decay as the value of H_0 decreases and, in the extreme case, a linear profile when H_0/W_0 is sufficiently small (as shown in Fig. 5.4). In principle, a condition that suppresses slip at the rough boundaries at the upper and the lower surface should also be important for justifying the stability of the profile. We may speculate whether the observed stick-slips in the experiment using a flat bottom (described in Section 5.1.2) can indeed be the consequence of not having a rough bottom that penalizes stick-slips and stabilizes the velocity profile near the bottom.

More generally, if the constraint of a monotonic increase for the function $f(\dot{\gamma})$ is lifted, a smooth velocity profile can be unstable and abrupt internal shear bands may occur (such as the case in some 2D particle dynamics simulations [1]). The measured velocity field in our system are all continuous, without showing discontinuities associated with internal shear bands. Whether this experimental fact sets a constraint on the possible rheological models for quasi-static grains, or is due to the difference between 3D flows and 2D flows, needs further clarification. Finally, we remark here that the simplistic theoretical pictures

[3] A direct comparison between the simple approximation Eqn (5.7) and the quadratic fit for the master curve shown in Fig. 5.4 give $\alpha = (0.103)^{-1} \cdot 2(\sigma_0^W/\sigma_0)(0.68mm/19.4mm)$, and $\beta = -2(0.0237)(0.103)^{-2} \approx -4$. The result for α is reasonable as long as σ_0^W is comparable to or smaller than σ_0, but the value for β is unsatisfactory. Further improvements of the approximation scheme, or of the simple assumptions made the internal rheology and sliding friction at sidewalls [Eqn (5.4) and Eqn (5.6)], may be in need.

[4] One possibility for establishing a boundary condition that suppress slip is to adapt Eqn (5.6) except for using a positive β at a rough surface.

discussed in this paragraph have not yet included any state variables that register the *history* of the packing, which we find essential for accounting the evolution of crystalline order.

It may also be interesting to see if one can find the similarity between the flow field of the sheared granular packing and that of a strongly *shear-thinning* fluid [5] in the same channel, since the latter may approach the case where shear stress has only a very weak dependence on the local shear rate. Furthermore, a disordered sheared packing may be compared to an isotropic fluid, while the crystallized flow clearly requires a model that exhibits anisotropy in its local rheology.

[5]Shear thinning is usually characterized by the behavior that, at high shear rate $\dot{\gamma}$, the shear stress $\sigma \propto \dot{\gamma}^n$ with $n < 1$; therefore the apparent viscosity $\eta \equiv \sigma/\dot{\gamma}$ decreases as the shear rate increases. (This covers a wide range of materials that can either become Newtonian ($n = 1$) at low shear rates or has a yield stress.) While an analogy to a granular flow may require $n \rightarrow 0^+$ (which is often coined as "plastic" material in the sense that its shear stress is insensitive to shear rate once it yields), real fluids that have an n below 0.4 are considered rare in practical circumstances.