

STRIVING FOR TRUTH
IN THE PRACTICE OF MATHEMATICS:
KANT AND FREGE¹

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Summary

My aim is to understand the practice of mathematics in a way that sheds light on the fact that it is at once a priori and capable of extending our knowledge. The account that is sketched draws first on the idea, derived from Kant, that a calculation or demonstration can yield new knowledge in virtue of the fact that the system of signs it employs involves primitive parts (e.g., the ten digits of arithmetic or the points, lines, angles, and areas of Euclidean geometry) that combine into wholes (numerals or drawn Euclidean figures) that are themselves parts of larger wholes (the array of written numerals in a calculation or the diagram of a Euclidean demonstration). Because wholes such as numerals and Euclidean figures both have parts and are parts of larger wholes, their parts can be recombined into new wholes in ways that enable extensions of our knowledge. I show that sentences of Frege's *Begriffsschrift* can also be read as involving three such levels of articulation; because they have these three levels, we can understand in essentially the same way how a proof from concepts alone can extend our knowledge.

Mathematics, the most venerable of the exact sciences, is also the most puzzling. How are we to understand a discipline that is, as mathematics seems to be, at once a priori and capable of yielding substantive knowledge, that is, judgments that are objectively true?² The most detailed, and in its way

1. This essay introduces a number of themes that are central to, and more adequately defended in, a much larger work-in-progress entitled *The Metaphysics of Judgment: Truth and Knowledge in the Exact Sciences*. What I aim to do here is only to sketch a way of thinking about the practice of mathematics that promises to resolve some of our most fundamental philosophical perplexities about that practice.

2. It was, of course, Kant who first formulated the problem, in the form of the question how synthetic a priori judgments are possible. A more recent formulation is due to Benacerraf (1973).

compelling, answer is Kant's: because intuition is involved in mathematical practice, according to Kant, mathematical practice can yield substantive knowledge; because the relevant intuition is pure, mathematical knowledge is nonetheless a priori. This answer is unacceptable, philosophically insofar as it entails transcendental idealism, and mathematically because, although it would have been less obvious to Kant in his time than it is to us in ours, at least some mathematics does not in any way involve the construction of concepts in pure intuition but only reasoning from concepts. How then are we to understand the striving for truth in mathematics? I will suggest that, with a little help from Kant, Frege's formula language of pure thought, his *Begriffsschrift*, provides the key.

Before Kant, logic was not conceived as something merely formal, that is, empty of all content.³ For Kant's predecessors, then, the idea that mathematical knowledge might be by means of reason alone did not entail that it is merely explicative. It is only in the context of Kant's own conception of cognition in terms of intuitions through which objects are given and concepts through which those objects are thought that there is a contradiction in the idea that one might, by reason alone, extend one's knowledge. For Kant, the fact that true mathematical judgments are ampliative, that they constitute real extensions of our knowledge, entails that they are not known by logic alone. Instead, he came to think, mathematical reasoning involves also appeal to constructions in pure intuition. The clue to understanding how this is to work is to be found in Kant's "Inquiry Concerning the Distinctness of the Principles of Natural Theology and Morality" written in 1764.

Kant (1764, p. 250) observes in the "Inquiry" that the mathematician, unlike the philosopher, "examines the universal under signs *in concreto*"; that is, instead of focusing on the thing itself, say time, as the philosopher does, the mathematician is most immediately concerned with signs. Beginning with signs for simple objects, the mathematician creates signs for distinct concepts of the objects of interest by combining those signs.⁴ In arithmetic and algebra, for example, "there are posited first of all not things themselves but their signs, together with the special designations of their increase or decrease, their relations *etc.*" (Kant 1764, p. 250).

3. See MacFarlane (2000).

4. A concept is distinct just if its characteristic marks and their relations one to another are clearly apprehended. Because in mathematics one creates definitions of one's objects through combining well-understood simples, according to Kant, even complex mathematical concepts are distinct.

Similarly, in geometry, one operates with diagrams, proving things about, say, triangles by considering one drawn triangle, that is, the concept *in concreto*. As a recent author has put what I take to be Kant's essential point here, "one doesn't speak mathematics but writes it. Equally important, one doesn't write it as one writes or notates speech; rather one 'writes' in some other, more originating and constitutive sense" (Rotman 2000, p. ix). Mathematics is *essentially* written.

But the interest of symbolism in arithmetic and algebra, and of ostensive constructions in Euclidean geometry, does not lie merely in the fact that it involves the manipulation of marks according to Kant. What is equally significant is just how, and why, this works. First, as Kant (1764, p. 251) remarks, signs in these contexts "show in their composition the constituent concepts of which the whole idea . . . consists". The Arabic numeral '278', for example, shows that the number designated consists of two hundreds, seven tens, and eight units. A drawn triangle similarly is manifestly a three-sided closed plane figure; like the numeral '278', it is a whole made up of simple parts. These complexes are then further combined to show "in their combinations the relations of the . . . thoughts to each other" (Kant 1764, p. 251).⁵ In mathematics, one combines the wholes that are created out of simples into larger wholes that exhibit relations among them. The systems of signs used in mathematics, then, have three distinct levels of articulation: first there are the primitive signs out of which everything else is composed; then there are the wholes formed out of those primitives, wholes that constitute the subject-matter of the relevant part of mathematics, the numbers of arithmetic, say, or the figures of Euclidean geometry; and finally there are the largest wholes (e.g., a Euclidean diagram, or a calculation in Arabic numeration) that are wholes of the (intermediate) wholes of primitive parts. It is just this feature of the systems of signs found in mathematics that is critical, I think, to mathematical practice as Kant understands it in the first *Critique*.

Consider Kant's famous example in the B Introduction, that of ' $7 + 5 = 12$ '. Leibniz had argued that such a truth is provable by means of logic and definitions alone, and before the critical period Kant had concurred.⁶ In the *Critique*, the view is rejected. Kant argues:

5. Kant is in fact describing what the words of natural language that are used in philosophy cannot do. It is clear that he means indirectly to say what the marks used in mathematics can do.

6. The evidence for Kant's early Leibnizean view is to be found in Kant (1980, 49–66), which is Herder's notes on Kant's lectures on mathematics. (See especially § 36.) Herder attended Kant's lectures from 1762 to 1764.

... the concept of twelve is by no means already thought merely by my thinking of that unification of seven and five, and no matter how long I analyze my concept of such a possible sum I will still not find twelve in it. One must go beyond these concepts, seeking assistance in the intuition that corresponds to one of the two, one's five fingers, say, or (as in Segner's arithmetic) five points, and one after another add the units of the five given in the intuition to the concept of seven ... I ... add the units that I have previously taken together in order to constitute the number 5 one after another to the number 7, and thus see the number 12 arise. (B15–16)

The number twelve, Kant suggests, must be constructed by the stepwise addition of units: a unit that is initially given as a part of the number 5 (defined as $4 + 1$) must come to be seen instead as a part of the given number 7 to yield thereby the number 8 (defined as $7 + 1$), and so on. The three levels of articulation that Kant identifies in the "Inquiry" are manifest. First there are the primitive parts, the units; next are the wholes of these parts, the given numbers seven and five conceived as collections of units; and finally we have the larger whole (potentially, the sum that is wanted) of which those given numbers are parts, the whole in virtue of which the parts of those numbers, the individual units, can be reconceived as parts of different wholes. Kant's point against Leibniz is that the fact that $7 + 1$ equals eight by definition cannot help in the demonstration until and unless we reconceive a unit given as a part of the number five instead as a part of a collection that includes the seven units of seven.⁷

According to Kant, essentially the same point can be made for the case of calculations involving larger numbers in the Arabic numeration system. Suppose that one wished to determine, say, the product of twenty-seven and forty-four. The Arabic numeration system provides the means as follows. First one writes signs for the two numbers to be multiplied in a particular array:

$$\begin{array}{r} 27 \\ \times 44 \\ \hline \end{array}$$

The first number is that given by the first line and the second is written directly below it. As in the case of the sum of seven and five as Kant conceives it, three levels of articulation are discernable, first, the primitive

7. Frege (1884, §6) makes a related point by appeal to the use of brackets in the formula language of arithmetic.

parts, that is, the primitive signs ‘2’, ‘4’, and ‘7’, then the wholes of these parts, namely the signs ‘27’ and ‘44’ that are signs for the numbers given as the terms of the problem, and finally, the whole display. The calculation is enabled by this three-leveled structure as follows. First, one reconfigures at the second level of articulation, taking a part of the whole ‘44’, namely the rightmost ‘4’, and reconceiving it as belonging with the ‘7’ in ‘27’. Multiplying the two numbers symbolized in this new whole yields the number twenty-eight so one puts a new primitive sign ‘8’ under the rightmost column and the sign ‘2’ above the leftmost column. Next one takes the same sign ‘4’ and considers it together with the ‘2’ in ‘27’, and so on in a familiar series of steps that result in the following:

$$\begin{array}{r}
 27 \\
 \times 44 \\
 \hline
 108 \\
 1080 \\
 \hline
 1188
 \end{array}$$

The last line is of course arrived at by the successive addition of the numbers given in the columns at the third and fourth rows. It is by reading down that one understands why just those signs appear in the bottom row; but it is by reading across that one knows the product that is wanted. In this way, through simple calculations on successive reconfigurations of various parts of the original display, one achieves the result that is wanted. Exactly the same point applies to calculations in algebra.⁸

In both the examples just considered, the three levels that Kant identifies in his “Inquiry” are discernable. But there is also an important difference between the two cases. In the first example, which we might think of as an instance of pebble arithmetic (as Frege would have called it), the task is merely to take a unit, that is, a stroke or pebble, first seen as a part of one whole to be a part of another. In the Arabic numeration system one does not merely reconfigure parts of wholes in this way; rather those reconfigura-

8. “Even the way algebraists proceed with their equations, from which by means of reduction they bring forth the truth together with the proof, is not a geometrical construction, but it is still a characteristic construction, in which one displays by signs in intuition the concepts, especially of relations of quantities, and, without even regarding the heuristic, secures all inferences against mistakes by placing each of them before one’s eyes”(A734/B762). One solves the problem by manipulating symbols “either through mere imagination, in pure intuition, or on paper, in empirical intuition, but in both cases completely a priori, without having to borrow the pattern for it from any experience” (A713/B741).

tions themselves set a task for the mathematician. One must add or multiply the numbers that are now taken together. To calculate in the Arabic numeration system, the array of numbers with which one begins must be seen now this way now that in a particular stepwise sequence that breaks the problem down into a series of simple steps. This difference between the two cases is, furthermore, a function of an essential difference between the way the signs operate in the two systems. Whereas in pebble arithmetic a stroke or a pebble is merely a unit that can be combined with other units in collections, the primitive signs of the Arabic numeration system, that is, the ten digits, have different meanings in different contexts. The Arabic numeration system is a positional system; what a digit means depends on the context in which it occurs (that is, on its position within the whole). In '14', for example, the digit '4' designates four, but in '41' it instead designates forty.⁹ Similarly, in our calculation above, when we multiplied the parts, we took the '2' in '27' to designate not two but twenty.

The positional Arabic numeration system is fundamentally different from pebble arithmetic, and from a system such as that of Roman numeration: whereas in the latter systems the primitive signs designate what they designate independent of the context of use (in Roman numeration, 'V' always means five, 'X' ten, and so on), in the Arabic system, what the primitive signs designate is determined only relative to a context of use, by their positions in the whole. Expressions involving both digits and signs for the basic arithmetic operations reinforces the point. Kant (1788, p. 283) explains in a letter to Schultz written shortly after the appearance of the second edition of the *Critique*:

I can form a concept of one and the same magnitude by means of several different kinds of composition and separation, (notice, however, that both addition and subtraction are syntheses). Objectively, the concept I form is indeed identical (as in every equation). But subjectively, depending on the type of composition that I think, in order to arrive at that concept, the concepts are very different. So that at any rate my judgment goes beyond the concept I have of the synthesis, in that the judgment substitutes another kind of synthesis (simpler and more appropriate to the construction) in place of the first synthesis, though it always determines the object in the same way.

9. Perhaps it will be objected that forty is just four tens, that is, that we should think of '41' as ' $10^1 \times 4 + 10^0 \times 1$ '. There are a number of problems with this line of thought, not least of which is the fact that it leaves us with no account of '10'. It also entails that calculations in Arabic numeration involve what are in fact quite complex algebraic manipulations when in fact they do not—as is shown by our example above.

Thus I can arrive at a single determination of a magnitude = 8 by means of $3 + 5$, or $12 - 4$, or 2×4 , or 2^3 , namely 8. But my thought “ $3 + 5$ ” did not include the thought “ 2×4 .” Just as little did it include the concept “8,” which is equal in value to both of these.

A concept, Kant suggests, can be considered either objectively or subjectively, and two concepts can be objectively the same, that is, concepts of one and the same object, but subjectively different. In ‘ 2×4 ’, for instance, the number eight is thought as a product; in ‘ $4 + 4$ ’ that same object is thought instead as a sum. It follows that in ‘ $4 + 4$ ’, say, the digit ‘4’ does not designate the number four; if it did, the whole could not be taken to designate eight, which Kant clearly thinks it does. Objectively, the two expressions ‘ 2×4 ’ and ‘ $4 + 4$ ’ are the same because in both cases the number designated is eight; but subjectively they are not the same because although what I think *of*, namely the number eight, is the same in the two cases, what I *think*, in the one case the sum of four and four and in the other the product of two and four, is quite different in the two cases. As Frege would put the point, while the two expressions designate one and the same object, they do so under different modes of determination; they express different senses.

In the “Inquiry”, Kant notes that arithmetic, algebra, and Euclidean geometry all employ systems of signs that involve three levels of articulation. By the time of the writing of the *Critique*, this insight is combined with his understanding of the distinction between intuitions through which objects are given and concepts through which (given) objects are thought to yield a much more detailed understanding of these mathematical practices. As Kant explains to Schultz, arithmetic expressions give objects under concepts in a way that requires that the primitives of the language to be variously interpretable depending on the context. What we need now to see is that just the same is true in Euclidean geometry.

Like the other systems we have considered, Euclidean geometry involves three levels of articulation. At the most basic level are the primitives of the system, the points, lines, angles, and areas, out of which everything else is constructed. At the second level are the objects we are interested in, those that form the subject matter of geometry, all of which are wholes of the primitive parts. At this level we find points as endpoints of lines, as points of intersection of lines, and as centers of circles; we find angles of various sorts that are limited by lines that are also parts of those angles; and we find figures of various sorts. A drawn figure such as (say) a square

has as parts: four straight line lengths, four points connecting them, four angles all of which are right, and the area that is bounded by those four lines. Of course, in the figure as actually drawn, the lines will not be truly straight or equal, and they will not meet at a point; the angles will not be right or all equal to one another. But this is immaterial because the drawn square is not merely a picture or instance of a square (any more than an Arabic numeral is merely a picture or instance of a number); it is a means of encoding information that is relevant to the demonstrations that will be made on the basis of it. Suppose, for instance, that one constructs a square on a line by appeal to the fact that all radii of a circle are equal, and the fact that given a line and a point not on the line one can draw a line through the point that is parallel to the given line. One then knows, on the basis of that construction, that the sides are all equal in length, that the angles are all right, and so on. It is the construction (together with the definitions) that determines what is true of the square, not what the drawn square looks like. And this is generally true of the figures that are discernable at the second level in Euclidean geometry.

At the third level, finally, is the whole diagram within which can be discerned various second-level objects depending on how one configures various collections of drawn lines (depending on how, as Kant would say, one synthesizes their manifold under a concept). As we will see, the cogency of Euclidean demonstrations crucially depends not only on our being able to see a given point, line, angle, or area now as a part of one figure and now as a part of another, but on our being able to see it as meaning something different in these different contexts, a given line, for instance, now as a radius of a circle and now as a side of a triangle.

The very first proposition in Euclid's *Elements* illustrates the essential point. The problem is to construct an equilateral triangle on a given finite straight line, and the demonstration is essentially as follows. First one constructs one circle with one endpoint of the given line as center and the line itself as radius, and another circle with the other endpoint as center and the line as radius. Then, from one of the two points of intersection of the two circles, one draws two lines, one to each of the endpoints of the original line. Now one reasons on the basis of the drawn diagram: two of the three lines are radii of one circle and so are known to be equal in length, and one of those radii along with the third line are radii of the other circle and so are known to be equal in length. But if the two lines in each of the two pairs are equal in length, and there is one line that is in both pairs, then all three lines must be equal in length. Those very same

lines, however, can also be conceived as the sides of a triangle. Because they can, we know that the triangle so constructed is equilateral.

The practice of mathematics is distinctive, Kant thinks, because it involves not merely reasoning from concepts, as in philosophy, but instead what Kant thinks of as the construction of concepts, either ostensively, as in Euclidean geometry, or symbolically, as in arithmetic, algebra, and higher analysis. It is through the mediation of constructions that one is able to discover truths that go well beyond what lay in the concepts with which one began. This works, I have suggested, in virtue of the three levels of articulation in these systems of signs. The objects of interest are at the second level but because they not only have parts but are themselves parts of a larger whole, their parts can be variously recombined and reconceived in ways that reveal new truths.

Unfortunately, even as Kant was working all this out, mathematicians were beginning more and more to eschew the sort of constructive approach that Kant focuses on and to adopt instead a conceptual approach, more and more, as one author puts it, “to conquer the problems with a minimum of blind calculation, a maximum of clear seeing thoughts” (Minkowski 1911 cited Stein 1988, p. 241).¹⁰ For example, whereas Euler (1748 cited in Boyer 1991, p. 443), one of the great masters of algebraic symbol manipulation, had defined a function of a variable quantity as “any analytic expression whatsoever made up from that variable quantity and from numbers or constant quantities”, that is as a kind of algebraic expression, Riemann (1826–1866), following Gauss (1777–1855), understands a function by way of its intrinsic properties. Instead of conceiving a function as an algebraic expression, Riemann focuses on the function’s “behavior”, whether or not the function is expressible algebraically.¹¹ Whereas early modern

10. Minkowski called this “the other Dirichlet Principle” in recognition of Dirichlet’s role in the development and implementation of this essentially new mathematical practice. Essentially new, but not completely unprecedented. It is not by diagrammatic reasoning that one knows that the square root of two cannot be expressed as a ratio of two whole numbers but by reasoning from concepts, and this was known even to the Pythagoreans.

11. Wilson (1992, p. 151, n. 42) provides this analogy: “build a bathtub and specify the ‘sources’ and ‘sinks’ where that water enters and leaves the tub. These conditions will completely fix how the water will flow in the rest of the tub. For Riemann a ‘complex function’ automatically corresponds to the ‘flow’ induced by its singularities, et al., whether or not there is any formula that everywhere matches such a flow”. This nineteenth century trend away from (constructive) problem solving and towards a more conceptual approach is evident also in Dedekind’s definition of the real numbers as “cuts”. Like Riemann, Dedekind’s dissertation director, and Gauss, with whom Dedekind habilitated, Dedekind (1932 cited Stein 1988, p. 245) aims to get at the concepts underlying various mathematical formulations, “to draw demonstrations, no longer

mathematics had focused on finding, that is, constructing, solutions to problems through the manipulation of algebraic expressions, mathematicians in Kant's time, and on through the nineteenth century, were more and more concerned to prove theorems on the basis of given definitions.

By the beginning of the twentieth century, two very different post-Kantian philosophies of mathematics had begun to emerge. The first and less radical approach, that which would come to dominate twentieth century thought, jettisons Kantian pure intuition, and with it the construction of concepts, but retains the Kantian conception of concepts as empty independent of relation to given objects and of logic as merely formal, that is, empty of all content. Because it was pure intuition that gave mathematics its content on Kant's view, it follows that without pure intuition mathematics must be understood in terms of the notion of logical form alone, that independent of any empirical interpretation mathematics must be conceived as a merely formal science without content or truth. It is but a small step to seeing that this conception of the propositions of mathematics is essentially that of standard model theory.

According to the standard model theoretic conception of language, a sentence, whether of natural language or of the symbolic language of arithmetic and algebra, is to be understood in terms of, on the one hand, its logical form, and on the other, its empirical content, if any, where that content is provided by an interpretation, or model or semantics that assigns a semantic value or designation to all the non-logical constants of the language. An axiomatization such as, for instance, Hilbert's of geometry, is taken to provide an implicit definition of the non-logical constants it involves, but only as to their logical form. Independent of an interpretation, which assigns a semantic value or designation to each primitive sign of the language, these axioms are neither true nor false. They are, as Kant would put it, thinkable independent of any relation to an object; but they can be cognized or known only in light of an interpretation that relates them to objects given in sensory experience.¹² There are, then, only two

from calculations, but directly from the characteristic fundamental concepts, and to construct a theory in such a way that it will, on the contrary, be in a position to predict the results of the calculation". In the case of the reals, Dedekind's aim is not to construct the numbers out of other numbers but instead to develop an adequate conception of them.

12. Friedman (1992, p. 55) outlines just this view for the case of geometry: "pure geometry is the study of the formal or logical relations between propositions in a particular axiomatic system, an axiomatic system for Euclidean geometry, say. As such it is indeed a priori and certain (as a priori and certain as logic is, anyway) but it involves no appeal to spatial intuition or any other kind of experience. Applied geometry, on the other hand, concerns the truth or falsity

possibilities: either mathematics belongs wholly to the realm of thought (that is, it is independent of any relation to objects given in intuition) in which case it is merely formal, empty, devoid of all content and truth; or it is contentful in virtue of its relation to ordinary material objects given in sense experience. Neither of these options is satisfactory. The first provides an account of the a priori character of mathematical practice but at the expense of an account of it as yielding substantive truths. The second, according to which mathematics concerns ordinary empirical objects, gives up on the a priori character of mathematical knowledge. The problem of understanding the striving for truth in mathematics that is set by this model theoretic conception of language in terms of logical form and empirical content is utterly intractable. We need another way. Frege, pre-figured by Bolzano, provides one. Rather than merely jettisoning Kantian pure intuition, Frege makes the more radical move of jettisoning *also* the Kantian conception of concepts according to which they are empty, mere forms, independent of relation to any object. Concepts, Frege argues, are fully objective entities in their own right, entities about which truths can be discovered independent of all reference to objects. In effect, Frege splits Kant's distinction between concept and intuition into two distinctions, that between *Sinn* and *Bedeutung*, on the one hand, and that between concept and object, on the other: objectivity is not inevitably grounded in relation to an object, and cognitive significance, *Sinn*, is not due to the involvement of concepts (see Macbeth (2005)). As a result, we will see, he is able to introduce a purely logical language that involves the same three levels of articulation that Kant had discerned in arithmetic, algebra, and Euclidean geometry. Because it has these three levels of articulation, Frege's *Begriffsschrift*, by contrast with our logical languages conceived model theoretically, enables reasoning from concepts that is at once a priori and ampliative.

We have seen that for Kant an arithmetic identity gives an object in two ways, or, as Kant thinks of it, as thought under two concepts. Frege, as already noted, thinks something similar. But, despite the impression one might get from reading "On Sense and Meaning", this cannot be the whole story for Frege. Consider, for example, the simple arithmetic identity ' $1 + 1 + 1 = 3$ ' that Frege discusses in *Grundlagen*. This sentence obviously

of such systems of axioms under a particular interpretation in the real world ... the truth (or approximate truth) of any particular axiom system [under an interpretation] is neither a priori nor certain but rather a matter for empirical investigation". Friedman aims here to be criticizing Kant but it is easy to see this view as essentially Kantian in its overall orientation.

expresses a truth of arithmetic. It presents, Kant thinks, a single object as thought under two concepts. If we instead read the sentence model theoretically, in terms of the idea that each primitive sign of the language designates prior to and independent of any context of use, then we will take each tokening of the numeral ‘1’ to designate the number one, and the sentence itself to express the fact (reminiscent of pebble arithmetic) that the number one and the number one and the number one together equal three. But as Frege argues in *Grundlagen*, that cannot be right: there is only one one and no amount of putting one together with itself will produce anything other than one.

There is only one one and yet the sentence ‘ $1 + 1 + 1 = 3$ ’ expresses a truth of arithmetic. We need to understand how this can be. Frege suggests the following. Instead of taking the primitive signs of the Arabic numeration system to designate numbers independent of any use—the numeral ‘1’, for example, to designate the number one whatever the context—we should understand such signs as only expressing a sense prior to their use. The signs are then put together to form a sentence that expresses a thought, and that thought can be analyzed into function and argument in various ways, none of which are privileged. The sentence ‘ $1 + 1 + 1 = 3$ ’ can be taken to involve, for instance, the function $\xi + 1 + 1 = 3$ with the number one as argument. Alternatively, we can, following Kant, take the object names ‘ $1 + 1 + 1$ ’ and ‘3’, both of which designate the number three (though they do so under different modes of presentation), to designate the arguments for the two-place relation $\xi = \zeta$; or alternatively, we can take the object name ‘ $1 + 1$ ’ to designate the argument for the function $\xi + 1 = 3$. Clearly other analyses are possible as well. So read the sentence does not present objects as thus and so. Instead it expresses a sense. Only relative to an analysis into function and argument are objects and concepts designated by the subsentential expressions of the language so conceived.¹³

13. Given this understanding of Frege’s mature conception of the workings of a symbolic language, it is easy to see that a well-known Dummettian criticism of Frege’s conception of the sense of predicate expressions is unfounded. The claim is that Frege puts two incompatible demands on the senses of predicate expressions: (1) that the compositionality of thought requires that the senses of predicate expressions be functions from the senses of singular terms to thoughts, which are the senses of sentences, and (2) that the senses of predicate expressions contain modes of presentation of functions from objects to truth-values. In fact, it is the senses of primitive expressions that are relevant to compositionality and the senses of those concept words that are the result of analysis that contain modes of presentation of functions from objects to truth-values. It is also, I think, easy now to see how there could be sentences that express thoughts though they fail to designate any truth-value. Consider the sentence ‘ $2/0 = 5$ ’. This

According to Frege's analysis in *Grundlagen*, as, in a different way, on Kant's, a sentence of arithmetic should be understood to involve not merely two levels of articulation as on the model theoretic conception but instead three, that of the primitive signs which are understood to express only a sense independent of their occurrence in a proposition, that of the sentence as a whole which expresses a thought and designates a truth-value, and between these two levels, that of the concept words and object names that are given relative to an analysis of the sentence into function and argument. Just the same is true of Frege's logical language *Begriffsschrift*. In Frege's logic we begin with primitive signs, the conditional and negation strokes, the concavity, and the content and judgment strokes, as well as two kinds of letters, Latin italic and German lending generality of content to sentences in which they occur. Frege's elucidations are to give us an understanding of what these signs mean, the senses they express, so that when we encounter them in a sentence we can grasp the thought that is expressed. Such sentences can then be analyzed into function and argument in various ways. As should be clear, this conception of language is quite different from that of standard model theory. On the model theoretic conception, we have seen, primitive signs designate independent of any context of use. Indeed, the interpretation or model or semantics is to serve precisely this purpose on the model theoretic conception; the interpretation, semantics, or model is to determine the (context-free) meanings or designations of the primitive expressions in a way that fixes the truth conditions of all sentences in which they occur. There are, then, only two levels of articulation in language conceived model theoretically, that of the primitive signs and that of whole sentences within which those signs occur.

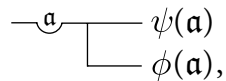
In Frege's logical language *Begriffsschrift*, I have suggested, primitive signs only express a sense independent of a context of use. Only relative to an analysis of a sentence into function and argument can we ask after the designations of various sub-sentential parts. It follows directly that one can form concept words of arbitrary complexity in Frege's logical language.

sentence is perfectly contentful; it expresses a thought that we grasp through our grasp of the senses of the primitive signs it involves. But this sentence does not designate any truth-value; it is neither true nor false because the sign '2/0' does not designate any number. Because that sign does not designate any number, there is no analysis of the sentence that yields expressions that succeed in designating an argument and a function. One or other of the relevant expressions will inevitably fail to denote anything. It follows that the sentence itself fails to denote any truth-value—despite the fact that it succeeds in expressing a thought.

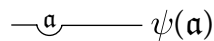
Consider, to begin with, the judgment that is expressed in *Begriffsschrift* thus:



This sentence, like any sentence of *Begriffsschrift*, can be variously analyzed. It can, for instance, be read as expressing the thought that the concepts (say) *red* and *colored* are related by subordination; it can be read, that is, as involving the second-level relation of subordination, designated by the expression



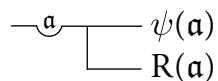
for arguments $C\xi$ and $R\xi$. This concept word formed out of the concavity with German letter and the conditional stroke is a concept word for the second-level relation of subordination; it designates that relation, and also, of course, expresses a sense. But we can also read our original sentence differently. For example, we can take it to involve the second-level concept word



designating the second-level property *universally instantiated*, for argument



Here we have a sign for a first-level conditional property: *being colored on the condition of being red*. This concept is, of course, universally instantiated; that is, it is a property of any object that might be given as argument for the function. We can also take our original sentence to involve the second-level concept



for argument Cξ. Clearly other analyses are possible as well.

We have seen that on Frege's mature conception of a sentence of *Begriffsschrift* three different levels of structure are discernable. At the lowest level a sentence is a collection of primitive signs of the language arranged in a certain way; at the highest level the sentence is itself a single unit, a whole that expresses a thought and designates a truth-value; and in between these levels are the object names and concept words that are revealed on an analysis. As has also been noted, a sentence on the model theoretic account has only two levels, that of its meaningful primitive parts (relative to a semantics or interpretation), parts that designate independent of any context of use in a sentence, and that of the whole that is made up of those parts. It follows that in Frege's logical language, though not in language conceived model theoretically, one can form concept words of arbitrary complexity, words that in the context of a whole sentence can be variously analyzed.

Consider the notion of continuity, which Weierstrass was the first clearly to articulate. As it is conceived quantificationally, Weierstrass's analysis provides not the content of the concept of continuity but instead the truth conditions of the claim that a function f is continuous at a point a : that claim is true if and only if

$$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x)(|x - a| < \delta \supset |f(x) - f(a)| < \varepsilon).$$

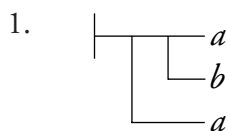
This sentence, as it is normally read, is composed of antecedently meaningful parts as specified in a standard semantics for the language, parts that are combined into a whole according to the syntactic rules of the language. Because it is so composed, the concept of continuity, on this account, has not being explained but reduced, that is, explained away. It is nothing over and above its parts in a given logical array.

Compare, now, that same concept as Frege (1880/81, p. 24) conceives it: that a function Φ is continuous at a point A is defined in *Begriffsschrift* as:

$$\begin{array}{l} \underbrace{\quad}_{\mathfrak{n}} \underbrace{\quad}_{\mathfrak{g}} \underbrace{\quad}_{\mathfrak{d}} \quad - \mathfrak{n} \leq \Phi(A + \mathfrak{d}) - \Phi(A) \leq \mathfrak{n} \\ \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad - \mathfrak{a} \leq \mathfrak{d} \leq \mathfrak{g} \\ \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \mathfrak{g} > 0 \\ \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \mathfrak{n} > 0 \end{array}$$

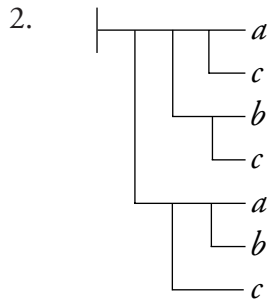
This sentence, read as Frege comes to read such sentences, only exhibits a sense, a Fregean thought, independent of any analysis—though in fact, in this case, an analysis is already indicated. We are to take the Greek letters ‘ Φ ’ and ‘ A ’ to mark the argument places for a higher-level concept word. What we have in this totality of signs, then, is an expression that designates (relative to the given analysis) the concept of continuity; we have a sign, a name, for this concept. The concept is not explained away, reduced to something else, on this account. It is instead *explained*; its cognitive content is exhibited in a way making it abundantly clear just what that content is.¹⁴ But of course, this content can, in the context of a sentence, be variously analyzed. It is precisely this feature of expressions in *Begriffsschrift* that will enable us to understand how a proof from concepts alone can be nonetheless fruitful, or ampliative, an extension of our knowledge. Much as a Euclidean diagram provides a medium within which to demonstrate the theorems of geometry by providing a means of exhibiting geometrical features of objects, and Arabic numeration provides us with a language within which to calculate by providing a means of exhibiting what we can think of as the computational contents of numbers, so Frege, we will see, provides us a language within which to reason by providing a means of exhibiting the cognitive contents of concepts.

Consider, first, a proof in *Begriffsschrift* of a simple theorem of logic, the rule that if a particular object has the property G but does not have the property F then it may be inferred generally that some G is not F. We begin with two axioms, though more will be added later, and one rule of inference, modus ponens. Given our one rule, it follows that the axioms and theorems of the system can serve in an inference in only two ways, either as the conditional or rule licensing the inference, or as marking the satisfaction of the condition, that is, as an instance to which the rule is to be applied. The two axioms, which following Frege we will understand as acknowledged truths, are:



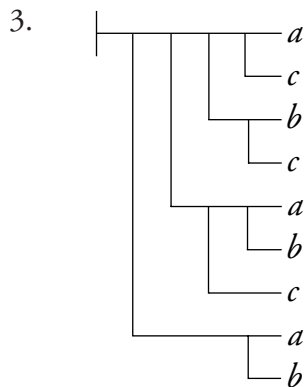
Axiom: If you know something then you know it on any condition. We can also read this axiom as saying that if a and b are true, then a is true.

14. This difference between the two sorts of language is developed in further detail in Macbeth (forthcoming).



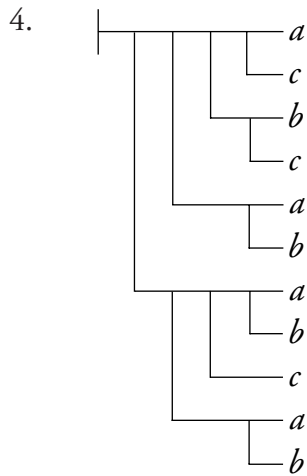
Axiom: If something (here, c) is a condition on a conditional (here, a -on-condition-that- b), then it is a condition of both condition and conditioned. Again we can analyze the sentence differently, for instance, as saying that if c and b imply a , and c implies b , then c implies a .

Suppose now that (2) is treated as satisfying the condition in (1), treated, that is, as an instance to which the rule expressed in (1) applies. It follows that we can put any condition we like on (2), given that it is true. That is, we treat all of (2), save for the judgment stroke, as standing in for 'a' in (1). For 'b' we can put anything we like, and then infer by modus ponens, with (1) as rule and (2) as instance, the relevant theorem. In particular, we can infer:



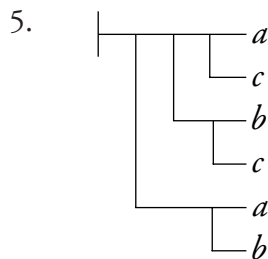
From (1) and (2): This is the result of putting (2) for the two occurrences of 'a' in (1) and the *Begriffsschrift* equivalent of 'a-on-condition-that-b' for 'b' in (1). Those substitutions give us a conditional of which (2) is the condition. (3) follows by modus ponens.

(2) tells us that if c is a condition on a conditional then it is a condition of both condition and conditioned. In (3) the first, or lowest, condition (a -on-condition-that- b) can be seen as a condition on a conditional, and so as satisfying the condition of (2). It therefore follows that:



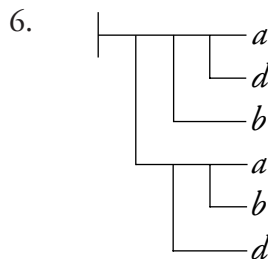
From (2) and (3): Here we have made the condition a-on-condition-that-b a condition of both condition and conditioned (given a particular analysis). This is justified by the rule expressed in (2) as applied to (3).

Notice now that the first, or lowest, condition of this sentence, (4), can be read as a substitution instance of (1); that is, it has, on one analysis, the logical form of (1), as can be seen by putting 'a-on-condition-that-b' for 'a' and 'c' for 'b' in (1). So the first (lowest) condition of (4) is satisfied and we can detach it to yield:



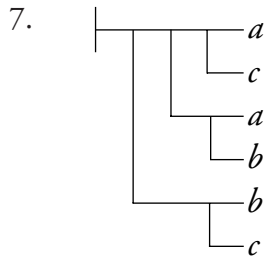
From (4) and (1): We can think of this sentence as telling us that if we have a conditional then we can attach any condition we like to both condition (i.e., b) and conditioned (i.e., a).

Now we need another axiom that says, in effect, that the order of conditions is immaterial.



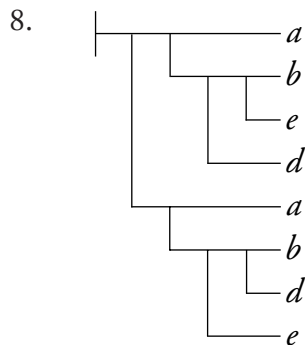
Axiom: If you have two conditions, b and d, on some sentence, then those conditions can be reversed.

So, by (6), we can reverse the order of the conditions in (5):



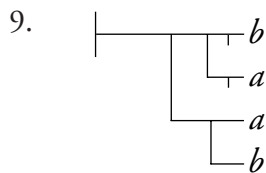
From (6) and (5): This gives as a formula the rule of hypothetical syllogism: if b-on-condition-that-c and a-on-condition-that-b, then a-on-condition-that-c.

Now we treat (6), which before we read as a rule, as instead satisfying the first, that is, lowest, condition in (7); that is, we treat the first (lowest) condition in (6) as the condition c in (7) and the rest of (6) as standing in for 'b', while changing letters appropriately (because otherwise different tokenings of 'a' would be playing different, yet unmarked, roles).



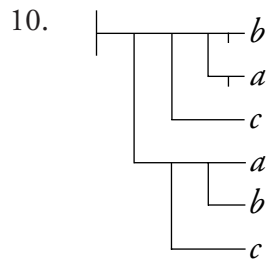
From (7) and (6): This seems intuitively to be true: if a on the condition that b-given-e-and-d, then a on the condition that b-given-d-and-e.

We introduce another axiom.



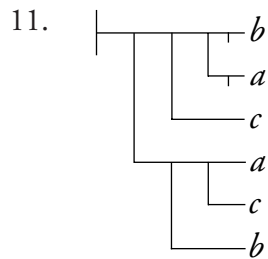
Axiom: This is just the rule of contraposition stated as a formula. It makes explicit the fundamental inferential significance of the negation stroke relative to the conditional stroke.

From (5) and (9), with (5) providing the rule and (9) the instance, that is, the condition that needs to be satisfied in (5), we can infer:



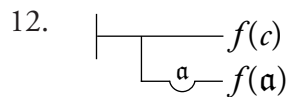
From (5) and (9): Here we have just added c as a condition on both the condition and the conditioned in (9). This is licensed by the rule expressed in (5).

(10), as instance, together with (8), as rule, yields:



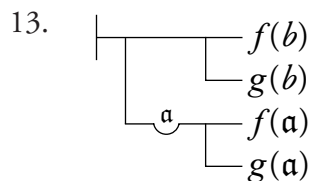
From (8) and (10): Licensed by (8), we have merely switched the order of 'b' and 'c' as they appear in the condition of (10). 'a' in (8), as that rule has been applied here, is everything to the right of the left-most conditional stroke, that is, all of the *Begriffsschrift* equivalent of 'not-b on condition that not-a and c'.

Again we introduce another axiom, that which makes explicit the basic inference potential of the concavity.



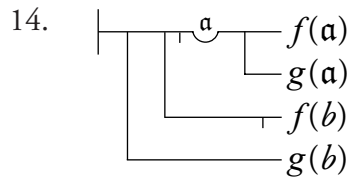
Axiom: If a property has the second-level property *universally instantiated* then it is a property of any particular thing.

But we want this axiom in a particular form, namely, as:



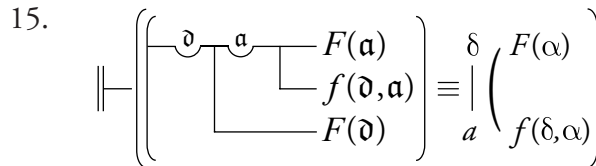
Instance of (12): Here we merely substituted a conditional property, being-f-on-condition-of-being-g, for the two occurrences of 'f' in (12).

From (13) as satisfying the condition on (11) conceived as a rule, finally, we get the theorem we set out to prove.

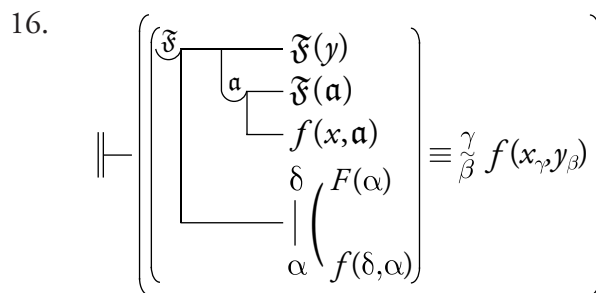


From (11) and (13): If some particular object b has the property g but not the property f then it can be inferred that some g is not f .

Now we need to see how things might work in mathematics, and for that we need some definitions. What we want to prove is that if an object x has some property that is hereditary in the f -sequence and y follows x in the f -sequence, then y has that property. To prove this we need definitions both of the concept *hereditary in a sequence* and of the concept *following in a sequence*.

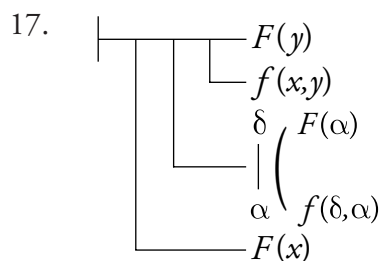


Definition of a property F being hereditary in the f -sequence, for some given f .



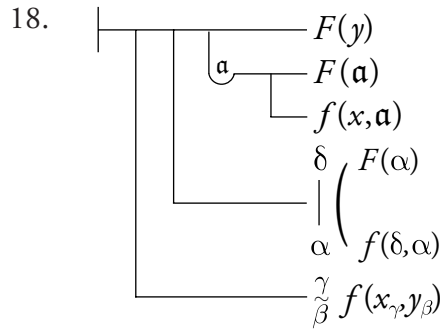
Definition of following in a sequence: y follows x in the f -sequence iff for all properties hereditary in the f -sequence if all objects bearing f to x have that property then y has that property.

Though we will not derive it formally, it should be relatively obvious that from (15) we can derive:



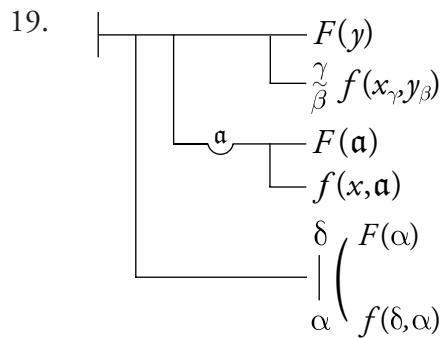
If x has the property F , and F is hereditary in the f -sequence, and y bears f to x , then y has the property F .

Similarly, from (16) we can derive the following conditional:



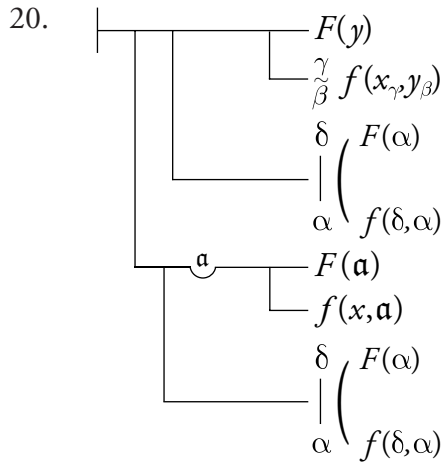
If y follows x in the f -sequence, then if F is hereditary in the f -sequence, and anything bearing f to x is F , then y has the property F .

Licensed by the rule in (6), we can switch around the three conditions in (18):



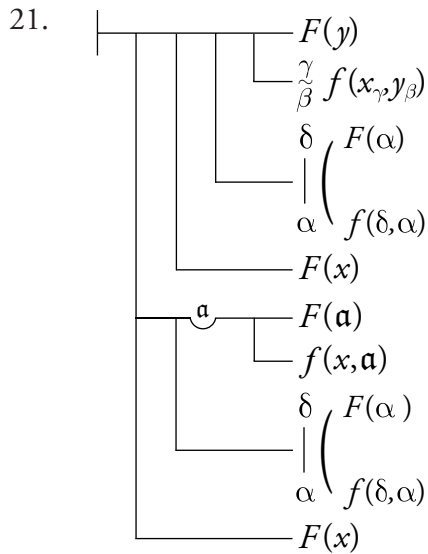
From (6) and (18): Here we have collapsed a couple of steps into one. The validity of the inference is obvious, and in *Grundgesetze* Frege introduces a rule licensing interchange of what he there calls subcomponents, that is, conditions.

Licensed by (2), we can see the lowest condition in (19) instead as a condition on both condition and conditioned (which is itself a conditional):



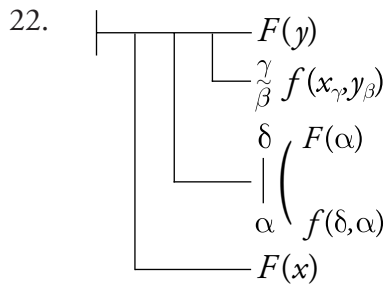
From (20 and (19): Not an especially pretty formula, but one should be able to see why it is true given that (19) is true. We should also by now have a pretty good idea of what is coming next.

Licensed by (5), we add a condition to both condition and conditioned in (20):



From (5) and (20): If it is true that any object bearing f to x has the property F if x has that property and that F is hereditary in the f -sequence, then it is true that y , which follows x in the f -sequence, has the property F if x has that property and it is hereditary in the f -sequence.

Because (17) provides an instance for the application of the rule expressed in (21), we can now infer the theorem that is wanted:



From (21) and (17): If x has the property F which is hereditary in the f -sequence then if y follows x in the f -sequence, y has the property F .

By appeal only to definitions and the axioms of pure logic, we have proved a theorem in the theory of sequences. There is, of course, an analogous proof in standard (second-order) quantificational logic. The point is not that one can prove, in a merely technical sense, something in Frege's logic, as if it might not be provable in that sense in our current logics; it is rather that given Frege's conception of language, and the essentially two-dimensional notation that puts that conception before our eyes, we can *see* how a purely logical proof can extend our knowledge. Much as in a Euclidean demonstration, one begins with some simple axioms and some definitions. But where Euclid also has postulates governing the construction of diagrams, we were able to work in the language itself. The sentences of *Begriffsschrift* that express our axioms and theorems are essentially two-dimensional, as of course Euclidean diagrams are, and because they are it is easy to see that they are variously analyzable, that they can be conceived in a variety of ways. In particular, a collection of primitive signs can be seen both as a single unit, as a concept word for some concept, and as providing a series of conditions on a judgment taken as a whole. Thus, we can consider the parts of a concept word as parts of different wholes and draw inferences on that basis. We can show, as we have seen, that given what it means to follow in a sequence or to be hereditary in a sequence, certain theorems are true. Of course here we have concepts that are purely logical, defined using only the primitive signs of logic. In mathematics, the concepts of interest will generally involve also primitive mathematical notions, for instance, that of a mathematical function in the definition of a group. The basic procedure, however, is the same: proof by appeal only to logic and definitions, where the proof essentially involves conceiving sentences now this way, now that.

But why are definitions, with their defined signs, needed at all in this process? Why not omit the definition and simply exhibit the full content of, say, the concept *following in a sequence* everywhere in the proof? After

all, if we did, everything would follow just as before. Everything would follow as before but it would be much harder to see the result as an interesting theorem in the theory of sequences, as something following from particular concepts. As Frege (1879, § 24) says, definitions “serve to call special attention to a particular combination of symbols from the abundance of possible ones”. Furthermore, such definitions, in the most interesting cases, are themselves the result of logical analysis. One begins with some familiar but as yet unanalyzed mathematical notion, say, that of being a prime number, or being a function continuous at a point; the task is to set out its precise content as it matters to judgment and inference, that is, “to articulate the sense clearly” (Frege 1914, p. 211). Having so articulated the sense, we “[obtain] a complex expression which in our opinion has the same sense” as the (unanalyzed) expression with which we began (Frege 1914, p. 210). Because, in *Begriffsschrift*, concept words are not one and all primitive signs, but can be formed out of combinations of primitive signs, we can express precisely that sense in Frege’s logical language, and then derive theorems from it. More specifically, we can consider parts of concept words now as parts of such (intermediate) wholes, now as parts of other (intermediate) wholes, and on that basis derive new truths. It is for just this reason that mathematics, on this account, is a substantive, albeit a priori, science. Indeed, so is logic.

Frege’s conception of language and of the nature of reasoning, I have suggested, is something essentially new. We have already seen that it is very different from the standard model theoretic conception; but it is also quite different even from the inferentialism of Sellars, and of Brandom following him. For Sellars and Brandom, the formal or purely logical rules of language must be supplemented with material rules, rules that are not formally valid but nonetheless license inferences. Conceptual contents, on their view, are inferential roles where “the inferences that matter for such content in general must be conceived to include those that are in some sense materially correct, not just those that are formally valid” (Brandom 1994, p. 105; see also Sellars 1953). For example, it follows logically, or formally, from something’s being red (all over) that it is colored because red just is a color of a certain kind; and it follows materially from something’s being red (all over) that it is not blue because although the concept *red* contains nothing involving being blue, nonetheless, the two are materially, that is, necessarily though not logically, incompatible. The concept involves, as Kant would put it, both what is analytic with respect to it and its synthetic a priori connections to other concepts. Grasping such a concept, then,

involves knowing both what follows formally from it and what follows materially from it. Concepts, on this view, have their own internal content and also necessary relations to other concepts, relations that might be made explicit in the axioms of a theory.¹⁵ Frege's conception of a concept is not inferentialist in this sense. For Frege the whole content of a concept is given by an adequate definition of it; no material rules are needed. The role of axioms is rather to set out the basic truths of the system, thoughts that can be seen to be true (assuming we have gotten things right) on the basis of our grasp of the senses expressed by the primitives of the language. And yet the judgments that can be derived on the basis of the definitions are, or at least can be, ampliative because, as we have seen, "often we need several definitions for the proof of some proposition, which consequently is not contained in any one of them alone, yet does follow purely logically from all of them together" (Frege 1884, § 88). This is exactly how one proceeds in abstract algebra, and in modern mathematics more generally.

Our problem was to understand the striving for truth in mathematics and in particular how the practice of mathematics can be at once a priori and capable of yielding substantive, objective truths. The solution came in two parts, one due to Kant and one due to Frege. From Kant we learned that a calculation or demonstration is ampliative in virtue of a fundamental feature of the systems of signs it employs, namely, its having three different levels or tiers of articulation: that of the primitive signs of the system, that of the largest whole of signs within which one works, and between the two both the signs for the objects that form the subject matter of the discipline and the various reconfigurations that enable one to see how, through a stepwise progression, one can establish the truth of the judgment one is interested in. What Kant did not have, and seems not to have thought it possible to have, was a system of signs that would serve in the expression, at the second level, of the contents of concepts. Frege provides exactly that in his formula language of pure thought. Because in *Begriffsschrift* primitive signs only express senses prior to their involvement in sentences, those sentences can be variously conceived in much the way the different parts of a Euclidean diagram can be variously conceived. The

15. Compare Hilbert's (1903, p. 51) remark in a letter to Frege, that "a concept can be fixed logically only by its relations to other concepts. These relations, formulated in certain statements, I call axioms, thus arriving at the view that axioms (perhaps together with propositions assigning names to concepts) are the definitions of the concepts". Axioms so conceived set out what Sellars and Brandom think of as the material inferential connections that are supposed to hold between concepts.

conception of language that is embodied in Frege's notation is not merely a conservative extension of Kant's insight, however. As I have indicated, Frege's logical language *Begriffsschrift* is something essentially new.¹⁶ It is also the language that is needed to understand the striving for truth in the practice of modern mathematics.

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16. Even Frege himself fully grasped the way his *Begriffsschrift* notation actually works only decades after its first introduction. See Macbeth (2005), especially Chapters Three and Four.

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