

Math 317 Fall 1998—Fixed Point In-Class
Project

Game Theory

Consider the following simple bluffing game, involving person A and person B. At the beginning of each round, each person puts in an ante of \$ a into the pot. The person A draws a card from an ordinary deck and looks at it. Now person A has two options: *fold* or *bet*. If A folds, then B wins the pot. If A bets, then A must put \$ b into the pot (the size b of this second bet is not up to A). Now person B has two choices: *fold* or *call*. If B folds, A wins the pot. If B calls, B must put \$ b in the pot, and then A shows the card. If the card is black, A wins the pot; otherwise, B wins the pot.

Pure strategies: Obviously, if the card drawn is black, A should always bet. The question is, should A bet or fold if a red card is drawn? A has two “pure strategies”: (1) bet all the time when drawing a red card or (2) fold all the time when drawing a red card. Similarly, B has two pure strategies: (1') fold whenever A bets or (2') call whenever A bets.

Payoff matrix: A's payoff matrix for this game is a 2-by-2 matrix M . m_{11} is A's expected gain if A uses pure strategy (1) and B uses pure strategy (1'); m_{12} is A's expected gain if A uses pure strategy (1) and B uses pure strategy (2'); etc. B's payoff matrix is the negative of A's payoff matrix, since whatever A wins, B loses, and vice versa (it's what's called a zero-sum game).

1. Compute A's payoff matrix for this game.

Each person knows the other's choice of strategy: The premise of our version of game theory is that each person knows the other person's choice of strategy. For example, if A picks strategy (1), then B will know that soon enough because A will never fold. Also, if A picks strategy (2), B will know that soon enough because A will bet half the time. Similarly, A can figure out after a few plays what B's choice of strategy is. So, the overriding assumption for our games is that each person knows exactly what strategy the other person is using.

2. Suppose A chooses strategy (1). What is B's most beneficial strategy? How much should B expect to win/lose with this combination of strategies? Suppose A chooses strategy (2). What is B's most beneficial strategy? How much should B expect to win/lose with this combination of strategies? So, if A assumes that B will pick the most beneficial strategy once he/she knows A's strategy, which strategy should A choose, and how much will A expect to win/lose in the game?

Mixed strategies: Rather than the pure strategies, it may be more beneficial to

use a probabilistic mix of strategies. For example, A could pick strategy (1) with probability x and strategy (2) with probability $1 - x$, with the choice in any given game determined randomly. The presumption is still that B knows what strategy A is using (i.e., B knows x) and will choose his/her own most beneficial mixed strategy, i.e. (1') with probability y , (2') with probability $1 - y$.

3. Simulate the game using a pair of mixed strategies, with $b = 2$, $a = 1$. Person A chooses $x = \frac{i}{6}$ for some $i = 1, 2, 3, 4, 5$ and announces it to B, who then guesses the most beneficial $y = \frac{j}{6}$ for $j = 0, 1, 2, 3, 4, 5, 6$. Run the game multiple times using the deck of cards and the dice (one die per person to generate random numbers for your strategy choices). Depending on your patience, try it at least 10 times, maybe 20 or more to get better statistics. If there's a Person C in your group, he/she can record the results to speed things up. Does A or B come out ahead? How do things change if $b = 4$ (a still = 1) – for this you don't need to replay the game, just recompute gains and losses from the previous simulation.

4. With the same x , choose a new y and rerun the simulation. What changes?

Equilibrium mixed strategies: Let's say there exist x_0, y_0 so that (a) when A chooses $x = x_0$, A's expected gain is $\geq \omega$ no matter what y is and (b) when B chooses $y = y_0$, A's expected gain is $\leq \omega$ no matter what x is. (It then follows that A's expected gain = ω if $x = x_0, y = y_0$). In some sense (x, y) is an equilibrium: it will not benefit A to choose $x \neq x_0$, because A can not hope to gain more than ω , since B can choose $y = y_0$ and keep A's gain $\leq \omega$. Similarly, it will not benefit B to choose $y \neq y_0$, because B can not force A to gain less than ω because A can choose $x = x_0$ and be sure of gaining $\geq \omega$.

5. Show that the existence of an equilibrium mixed strategy boils down to a solution (x_0, y_0) of the inequalities:

$$\begin{bmatrix} x_0 & 1 - x_0 \end{bmatrix} M \geq \begin{bmatrix} \omega & \omega \end{bmatrix}, \quad M \begin{bmatrix} y_0 \\ 1 - y_0 \end{bmatrix} \leq \begin{bmatrix} \omega \\ \omega \end{bmatrix}.$$

(Vector inequalities mean each component of the vector satisfies the inequality).

6. Solve the above inequalities for x_0, y_0, ω for your matrix M . How do they depend on b/a and why does that make sense?

John von Neumann proved that for any payoff matrix, there is always an equilibrium mixed strategy. How to compute it is an issue for another course (Linear Optimization). By the end of this course, we will prove that it exists (and similarly for more complicated games).

A *contraction mapping* is a continuous function $f : S \rightarrow S$, (S a closed subset of a normed vector space V) for which there exists a real number θ , $0 \leq \theta < 1$, so that $\|f(x) - f(y)\| \leq \theta\|x - y\|$ for all $x, y \in S$.

The goal is to prove the following:

Contraction Mapping Theorem: A contraction mapping has a unique fixed point; i.e. there is a unique $x^* \in S$ with $f(x^*) = x^*$.

1. Choose any $x_1 \in S$ and build a recursive sequence $x_{n+1} = f(x_n)$. Prove the recursive sequence converges (think about the name of the Theorem), and then prove its limit is a fixed point.

This proof also provides a way to compute the fixed point: the *method of successive approximations*. You choose any point in S , and compute the recursive sequence until it appears to have converged to a suitable degree.

2. Prove uniqueness.

Assume for the rest that $S \subseteq \mathbb{R}$.

3. Show that if a contraction mapping is differentiable, then $|f'(a)| \leq \theta$ for all $a \in S$. Show that if $|f'(a)| \leq \theta$, then f is a contraction mapping in a neighborhood of a .

4. On HW, you showed that $\cos x = x$ has a solution for some $x \in [\pi/6, \pi/3]$. Show that $f(x) = \cos x$ is a contraction mapping in a neighborhood of that root using (3). (*Optional: Show that in fact f is a contraction mapping on all of $[\pi/6, \pi/3]$*). Compute the solution on your calculator by the method of successive approximations, using a reasonable number of steps.

A famous (at least local) contraction mapping plays a role in Newton's method: to find a root x^* of $g(x)$ (i.e., x^* so that $g(x^*) = 0$), we consider the function $f(x) = x - \frac{g(x)}{g'(x)}$.

5. Show that x^* is a fixed point of f if and only if x^* is a root of g . Using (3), argue that f is a contraction in the neighborhood of the root, and that in fact $\theta \rightarrow 0$ near x^* .

6. Compute the solution to $\cos x = x$ on your calculator using Newton's method, using the same number of steps as in (4). How does the rate of convergence compare? Why is that?

The Hairy Billiard Ball

Let $S^2 = \{\vec{x} = (x_1, x_2, x_3) \in \mathbb{R}^3 : \|\vec{x}\| = \sqrt{x_1^2 + x_2^2 + x_3^2} = 1\}$ be the standard sphere in \mathbb{R}^3 .

A unit-tangent-vector-field on S^2 is a function $\vec{v} : S^2 \rightarrow \mathbb{R}^3$ so that $\|\vec{v}(\vec{x})\| = 1$ and $\vec{v}(\vec{x}) \perp \vec{x}$, i.e. $v_1(\vec{x})x_1 + v_2(\vec{x})x_2 + v_3(\vec{x})x_3 = 0$.

The Hairy Billiard Ball Theorem says there is no continuous unit-tangent-vector-field on S^2 . You will prove that there is no unit-tangent-vector-field with continuous derivative on S^2 (and we will discuss how to extend the proof to all continuous unit-tangent-vector-fields in class). It's a proof by contradiction, so you start by assuming \vec{v} with continuous derivative exists.

1. We need the fact that there is some constant $\lambda > 0$ so that $\|\vec{v}(\vec{x}) - \vec{v}(\vec{y})\| \leq \lambda\|\vec{x} - \vec{y}\|$ for all $\vec{x}, \vec{y} \in S^2$. Proving this involves some multivariable stuff that we haven't covered yet, so you will just prove the \mathbb{R}^1 version: If $f : [a, b] \rightarrow \mathbb{R}$ has a continuous derivative $f'(x)$, prove there is some constant $\lambda > 0$ so that $|f(x) - f(y)| \leq \lambda|x - y|$ for all $a \leq x, y \leq b$. One way to do this is to show that the following function $w : [a, b] \times [a, b] \rightarrow \mathbb{R}$

$$w(x, y) = \begin{cases} \frac{f(x) - f(y)}{x - y} & \text{for } x \neq y \\ f'(x) & \text{for } x = y \end{cases}$$

is continuous on its compact domain, hence bounded. Observe that the continuity of w is not trivial because the definition of $f'(x)$ only assures continuity of w along vertical paths approaching x in $[a, b] \times [a, b]$. Use the fact from class that $f'(x)$ is *uniformly* continuous on the compact set $[a, b]$ to construct a proof of w 's continuity.

We now extend \vec{v} from S^2 to an annulus

$$A = \{\vec{x} \in \mathbb{R}^3 : \frac{1}{2} \leq \|\vec{x}\| \leq \frac{3}{2}\}$$

by the definition $\vec{v}(r\vec{x}) = r\vec{v}(\vec{x})$ for $\frac{1}{2} \leq r \leq \frac{3}{2}$ and $\|\vec{x}\| = 1$. Note that for $\vec{x} \in A$, $\|\vec{v}(\vec{x})\| = \|\vec{x}\|$.

Now, for any real number $t < \max(\frac{1}{3}, \frac{1}{\lambda})$, define the function $\vec{g}_t : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ by $\vec{g}_t(\vec{x}) = \vec{x} + t\vec{v}(\vec{x})$.

2. Show using the perpendicularity of \vec{x} and $\vec{v}(\vec{x})$ that $\|\vec{g}_t(\vec{x})\| = \sqrt{1 + t^2}\|\vec{x}\|$.

Thus, $\vec{g}_t : A \rightarrow A_t$, where

$$A_t = \left\{ \vec{x} \in \mathbb{R}^3 : \frac{\sqrt{1+t^2}}{2} \leq \|\vec{x}\| \leq \frac{3\sqrt{1+t^2}}{2} \right\}$$

In another part of this project, the Contraction Mapping Theorem was proven: A *contraction mapping* is a continuous function $f : S \rightarrow S$, (S a closed subset of a normed vector space V) for which there exists a real number θ , $0 \leq \theta < 1$, so that $\|f(x) - f(y)\| \leq \theta\|x - y\|$ for all $x, y \in S$.

Contraction Mapping Theorem: A contraction mapping has a unique fixed point; i.e. there is a unique $x^* \in S$ with $f(x^*) = x^*$.

3. For $\vec{c} \in S^2$, prove there is a $\vec{x} \in A$ so that $\vec{g}_t(\vec{x}) = \vec{c}$. (Use the Contraction Mapping Theorem on $\vec{f}(\vec{x}) = \vec{c} - t\vec{v}(\vec{x})$). Show that $\|\vec{x}\| = \frac{1}{\sqrt{1+t^2}}$.

4. Show that for any $\vec{c} \in A_t$, there is an $\vec{x} \in A$ so that $\vec{g}_t(\vec{x}) = \vec{c}$. (Use (3) to show that there exists $\vec{x}_1 \in A$ with $\vec{g}_t(\vec{x}_1) = \frac{\vec{c}}{\|\vec{c}\|}$. Then let $\vec{x} = \|\vec{c}\|\vec{x}_1$ and show that $\vec{g}_t(\vec{x}) = \vec{c}$ and that $\vec{x} \in A$).

Thus, $\vec{g}_t : A \rightarrow A_t$ is onto.

5. Compute the volume of A_t as a difference of spheres; this is not a polynomial function of t .

6. Compute the volume of A_t by integrating the Jacobian of \vec{g}_t over A ; this is a polynomial function of t .

Thus we have a contradiction.

Topological Equivalence

We say two sets S_1, S_2 are topologically equivalent if there exists $g : S_1 \rightarrow S_2$ continuous and onto that has a continuous inverse $g^{-1} : S_2 \rightarrow S_1$.

Let $B^n \in \mathbb{R}^n$ be the standard closed unit ball:

$$B^n = \{\vec{x} \in \mathbb{R}^n : \|\vec{x}\| \leq 1\}.$$

In another part of the project, we prove the Brouwer Fixed Point Theorem: If $f : B^n \rightarrow B^n$ is continuous, then $f(\vec{x}) = \vec{x}$ for some $\vec{x} \in B^n$.

1. Prove, using the Brouwer Fixed Point Theorem on B^n , that if a set S is topologically equivalent to B^n , and $h : S \rightarrow S$ is continuous, then $f(y) = y$ for some $y \in S$.

Let $S \in \mathbb{R}^n$ be a closed, convex, bounded set with an interior point \vec{b} . You will prove that S is topologically equivalent to B^n (so, the Brouwer Fixed Point Theorem will hold on S).

2. Prove that for all $\vec{x} \in \mathbb{R}^n$ with $\|\vec{x}\| = 1$, there exists a real number $r(\vec{x})$ so that $\vec{b} + \lambda\vec{x} \in S$ for $0 \leq \lambda \leq r(\vec{x})$ and $\vec{b} + \lambda\vec{x} \notin S$ for $\lambda > r(\vec{x})$.

3. Prove that $\vec{b} + r(\vec{x})\vec{x} \in bd(S)$.

4. Prove that $r(\vec{x})$ is continuous. (If not continuous at some \vec{x}_0 , there would be a Cauchy sequence of \vec{x}_k with $\|\vec{x}_k\| = 1$ converging to \vec{x}_0 , but $r(\vec{x}_k)$ would not converge to $r(\vec{x}_0)$. However, $\{r(\vec{x}_k)\}$ is a bounded sequence, so has a convergent subsequence $\{r(\vec{x}_{k_j})\}$. Consider the subsequence $\vec{b} + r(\vec{x}_{k_j})\vec{x}_{k_j}$.)

5. Given $\vec{z} \in B^n$, let

$$g(\vec{z}) = \begin{cases} \vec{b} + r\left(\frac{\vec{z}}{\|\vec{z}\|}\right)\vec{z} & \text{for } \vec{z} \neq 0 \\ \vec{b} & \text{for } \vec{z} = 0 \end{cases}$$

Show that $g : B^n \rightarrow S$ is continuous and onto.

6. Show that

$$g^{-1}(\vec{y}) = \begin{cases} \frac{\|\vec{y}-\vec{b}\|}{r(\vec{x}(\vec{y}))}\vec{x}(\vec{y}) & \text{for } \vec{y} \neq \vec{b} \\ 0 & \text{for } \vec{y} = \vec{b} \end{cases},$$

where $\vec{x}(\vec{y}) = \frac{\vec{y}-\vec{b}}{\|\vec{y}-\vec{b}\|}$. Show that $g^{-1} : S \rightarrow B^n$ is continuous.