Math 222 Fall 2002—Lab #5 Due: Fri. Dec. 13

Reading: C& K 12.1 (not "Modifications and Refinements"), 12.2, Ch. 13 through Section 13.2

Assignment: Do one of the two "Boundary Value Problem" choices and one of the two "PDE" choices. See `www.haverford.edu/math/rmanning/math222/hw.html` for sample notebooks demonstrated in class which you may use as templates. Be sure to read and understand my commands first!

Boundary Value Problem Choice 1: Shortest Trip Down a Wire

The following integral gives the time of travel for a bead falling down a wire with shape $y(x)$ beginning at $(0, a)$ with initial velocity 1 and ending at $(b, 0)$:

$$t_{fall} = \int_0^b \frac{\sqrt{1+(y')^2}}{\sqrt{2g(a-y)+1}}dx.$$

(I made the initial velocity 1 rather than 0 as we did in class because the integral is improper for the initial-velocity-0 problem, and we don't know how to handle that).

If one cranks through the Euler-Lagrange theory for finding the $y(x)$ that minimizes t_{fall} subject to $y(0) = a$ and $y(b) = 0$, one gets the ODE:

$$y'' = \frac{g(1+(y')^2)}{2g(a-y)+1}$$

Adapt my Mathematica implementation of shooting to solve this problem, for $a = b = 1$ and $g = 10m/s^2$. What is $y'(0)$ for the optimal shape? Graph the optimal $y(x)$, and compute the corresponding value of t_{fall}.

Boundary Value Problem Choice 2: Spending Your Nest Egg

Imagine that you are retiring today and you expect to live for 20 years. You have \$ 500000 in the bank to support yourself for those 20 years. Let $w(t)$ be the amount (in thousands of dollars, to make the numbers nicer) you have in the bank at time t. Then one model for the total present value (after taxes) of all your withdrawals from the bank is

$$PV = \int_0^{20} e^{-rt} f(Rw - w')dt,$$

where

$$f(x) = x\left(\frac{1}{2} + \frac{1}{2}e^{-(x/50)^2/2}\right).$$

(If you're interested, the model assumes an interest rate of R from the bank and an inflation rate of r. $f(x)$ is the amount you have after taxes if you withdraw x in some year; if you take out $x \approx 0$, then $f(x) \approx x$ since you pay almost no tax, whereas if x is large, then $f(x) \approx x/2$, since you lose about half of your money to taxes).

If one cranks through the Euler-Lagrange theory for finding the $w(t)$ that maximizes PV subject to $w(0) = 500$ and $w(20) = 0$, one gets the ODE:

$$w'' = Rw' - (r - R)\frac{f'(Rw - w')}{f''(Rw - w')}.$$

Adapt my Mathematica implementation of shooting to solve this problem, for $r = 0.06$ and $R = 0.08$. What is $w'(0)$ for the optimal spending plan? Graph the optimal $w(t)$ and compute the corresponding PV

PDE Choice 1: Black-Scholes Model for Pricing an Option

In 1997, F. Black, M. Scholes, and R. Merton won the Nobel Prize in Economics for their model for the pricing of "options". This model is a PDE much like the heat equation.

Roughly, the way (one type of) option works (a "put option") is that you own 1 share of a stock today that is worth \$ S, and you and I make a deal that t years from now you will be given the choice of selling me that stock for some fixed price \$ E (and I *must* buy it from you at that price, no matter what the stock is worth on that day). Of course, if t years from now the stock's price on the market is more than \$ E, you will not choose to sell it to me for \$ E.

The question is: what is a fair price for you to pay me today for this option to sell t years from now? This "fair price" is called the price of the option, which we will call $V(S,t)$. As indicated in the notation, we think of V as depending primarily on two variables: the current stock price S and the time until the option can be exercises t (although V also depends on a few other parameters, like E). The option price satisfies the *Black-Scholes equation*:

$$\begin{aligned}
\frac{\partial V(S,t)}{\partial t} &= \frac{1}{2}v^2 S^2 \frac{\partial^2 V(S,t)}{\partial S^2} + rS\frac{\partial V(S,t)}{\partial S} - rV(S,t). \\
V(0,t) &= Ee^{-rt} \text{ for all } t \\
V(S_{max},t) &= 0 \text{ for all } t \\
V(S,0) &= \begin{cases} E-S & \text{if } E > S \\ 0 & \text{if } E \leq S \end{cases}
\end{aligned}$$

(The parameter r is the annual interest rate, and v measures the stock price's volatility. The second equation is the present value of a payment E obtained t years from now, since if $S = 0$ now, it will remain at $S = 0$ and you will get \$ E at the sale. The third equation says that if the stock currently has the very high price S_{max}, we assume we will not exercise the option as it will not fall below \$ E in t years. The fourth equation says that if the option were exercised right away, you would gain $E - S$ if the deal price E is greater than the stock price S.)

Adapt my Mathematica implementation of the explicit solution of the heat equation to solve the given Black-Scholes equation. Use $E = 95$, $S_{max} = 130$, $v = 0.75$, and $r = 0.05$. *Note: Because the PDE is different from the standard heat equation, the instability cutoff for* $\frac{k}{h^2}$ *is no longer* $\frac{1}{2}$. *Experiment with choices of k until you find stability.* Take enough steps to reach $t = 4$ years. What is the price of the option if $S = 75$ and $t = 2$? Make a movie out of the graphs of $V(S,t)$ versus S for $t = 0, 0.5, 1, 1.5, 2, 2.5, 3, 3.5, 4$. Explain how the qualitative change in the graph over time matches intuitively how the value of an option "should" change. Please submit the Mathematica notebook to me by email or on a disk so that I can view the movie.

PDE Choice 2: Chemical Spill revisited

The Perilites, after years of bad experiences with spills from the nearby ACME chemical plant, have developed a more sophisticated model for the spread of chemical waste. They have determined that the following PDE is a reasonable model for the benzene concentration $u(x, y, t)$ (in liters per square mile) as a function of x and y positions and time t:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - pue^{-(x-4.5)^2/2-(y-3.5)^2},$$

where the last term models the removal of benzene by a pumping station centered at $(x, y) = (4.5, 3.5)$, with the parameter p representing the level of pumping.

At time 0, the concentration is

$$u(x, y, 0) = \begin{cases} 700e^{1/(x^2+y^2-4.1)} & \text{for } x^2 + y^2 < 4.1 \\ 0 & \text{for } x^2 + y^2 \geq 4.1, \end{cases}$$

a 2D "bump" function centered at the ACME plant at the origin, decaying smoothly to zero at the circle $x^2 + y^2 = 4.1$ (I made it 4.1 so you would be unlikely to have to worry about the denominator of the exponent vanishing on your discretization).

The Perilites have chosen the follow boundary conditions for their model: $\frac{\partial u}{\partial x} = 0$ at $x = \pm 8$ and $\frac{\partial u}{\partial y} = 0$ at $y = \pm 8$. (Clearly they do not want $u = 0$ on the boundary, since waste will eventually diffuse to that point; this sort of first-derivatve "Neumann" condition is a common alternative)

Adapt my Mathematica implementation of the explicit solution of the 2D heat equation to solve this PDE system. For $p = 2$, compare $u(6, 6, t)$ to $u(-6, -6, t)$ for a length of time that shows the arrival fo the waste and part of its eventual decay (recall that $(6, 6)$ is the center of Peril; $(-6, -6)$ is the center of Peril's archrival city, Southwest Peril.) *Note: Because of the difference between this PDE and the standard heat equation, the instability cutoff for* $\frac{k}{h^2}$ *is no longer* $\frac{1}{2}$. *Experiment with choices of* k *until you find stability.* You will need to discretize the boundary conditions in an appropriate way. Make a movie showing the graph of $u(x, y, t)$ for (x, y) in the square $-8 \leq x, y \leq 8$ for 25 times in your total time interval. Please submit the Mathematica notebook to me by email or on a disk so that I can view the movie.