Math 222 Fall 2002—Lab #1 Due: Fri. Sep. 27

Reading (from Cheney and Kincaid): §3.2 through p. 112. Proof of quadratic convergence on pp. 107–108 may be skipped.

Assignment: Do one of the three choices labeled "1-variable Newton's Method" and one of the three choices labeled "n-variable Newton's Method"

1-variable Newton's Method Choice 1: Kepler's Equation

As we saw in class, Kepler's equation gives the angular position x of a body E orbiting in an ellipse about some star S at one focus of the ellipse. The equation is

$$\frac{2\pi t}{T} = x - e\sin x,$$

where T is the period of motion, t is the time after closest approach of E to S, e is the eccentricity of the orbit, and x is the angle of the body with respect to the center of the ellipse. Choose units of time so that $T = 1$ (if E is the earth, this means we measure time in units of years; otherwise, we measure time in units of periods of revolution).

For the earth's eccentricity $e = 0.0167$, use Newton's method to find the value of x corresponding to $t = 0.25$, 0.5, and 0.75 (take as many steps as you need to reach acceptable convergence).

Use Mathematica to construct a function `x[t_,e_]` that will compute x for given values of t and e by running Newton's method. For $e = 0.0167$, plot this function to get a graph of x as a function of t for $0 \le t \le 1$. Describe what you see in terms of the earth's motion. Repeat this for $e = 0.25$ and $e = 0.75$. What changes?

1-variable Newton's Method Choice 2: A Polluted River

A healthy river has a certain amount of dissolved oxygen, while a polluted river can have an oxygen deficit D (how much less oxygen the river has than normal, measured in mg/L). This deficit arises because the waste in the river uses up a certain amount of oxygen W (measured again in mg/L) as it is consumed by bacteria in the water.

Under the simple Streeter-Phelps model, these two quantities evolve in time according to the pair of differential equations:

$$\begin{aligned}\frac{dW}{dt} &= -k_d W\\ \frac{dD}{dt} &= k_d W - k_r D\end{aligned}$$

where k_d is a rate constant for the decay of the waste and k_r is a rate constant for the aeration of the river by outside oxygen.

(a) Explain the physical meaning for each of the three terms on the right hand side ($-k_d W$, $k_d W$, and $-k_r D$), including why their signs make sense.

(b) Verify (either by hand or by plugging into Mathematica), that the following pair of functions is a solution to the above differential equation:

$$\begin{aligned} W(t) &= W_0 e^{-k_d t} \\ D(t) &= \frac{k_d W_0}{k_r - k_d}\left(e^{-k_d t} - e^{-k_r t}\right) + D_0 e^{-k_r t} \end{aligned}$$

(Here W_0 is the initial concentration of the waste and D_0 is the initial oxygen deficit).

Assume that $D_0 = 2$ mg/L, $W_0 = 20$ mg/L, $k_d = 0.25$ (days)$^{-1}$, and $k_r = 0.33$ (days)$^{-1}$.

(c) Graph $D(t)$ for $0 \leq t \leq 30$. Using Newton's method, find the time at which the oxygen deficit returns to its initial level. Using Newton's method, find the time at which the oxygen deficit reaches the level of 0.1 mg/L.

1-variable Newton's Method Choice 3: Profit Maximization

You own ACME Auto Company. Market research indicates that at a price of \$ 10000, you will sell 50 million cars, while at higher prices, this demand falls linearly until it hits zero at price \$ 50000. Use this information to find the linear function $p(x)$ relating price p to demand x.

If you make x cars, the (marginal) cost to produce an additional car is $C'(x) = 10000e^{-x/20{,}000{,}000} + 10000$. Based on this, find the total cost $C(x)$ of producing x cars.

The government taxes you t dollars per car produced. Thus, your total profit for producing x cars is:

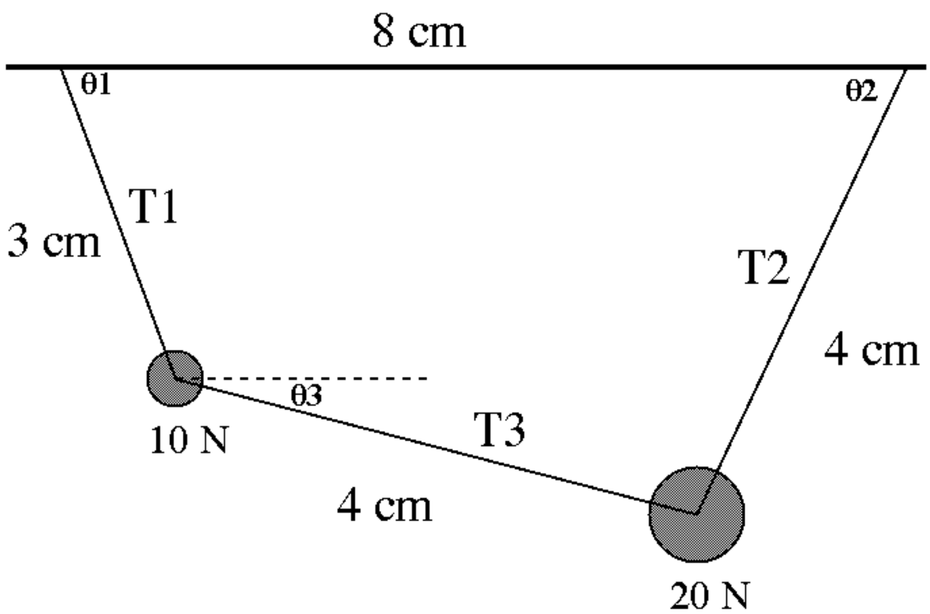
$$Profit = xp(x) - C(x) - tx.$$

To maximize profit, differentiate this and set it equal to zero. Use Newton's method to find the value of x that maximizes the profit when $t = 4000$.

Use Mathematica to construct a function `x[t_]` that will compute the optimal value of x for a given value of t by running Newton's method. Now imagine you are not the ACME owner, but the government. For any tax rate t, assume that ACME will set its production level x to maximize profit. Using `x[t_]`, estimate the value of t that will maximize the tax revenue the government receives from ACME.

Multivariable Newton's Method Choice 1: Mechanical Equilibrium

Consider the system of ropes and weights below. There are 6 unknowns: three tensions T_1, T_2,



and T_3, and three angles θ_1, θ_2, and θ_3. Find 6 equations relating them: x and y force balance equations on the 10-Newton weight (tensions always pull on the masses), x and y force balance equations on the 20-Newton weight, and the two spatial constraints:

$$\begin{aligned} 3\cos\theta_1 + 4\cos\theta_2 + 4\cos\theta_3 &= 8 \\ 3\sin\theta_1 - 4\sin\theta_2 + 4\sin\theta_3 &= 0 \end{aligned}$$

Use Newton's method to find $(T_1, T_2, T_3, \theta_1, \theta_2, \theta_3)$.

Multivariable Newton's Method Choice 2: Resource Allocation

You run ACME Robot Corporation, and you have a sales target of 14 thousand robots over the next year. You can hire salespeople at \$30000 each per year or technicians at \$50000 each per year. Internal research shows that as a function of the number of salespeople x and number of technicians y, the number of robots you will sell will be:

$$S(x,y) = \frac{(15000 - 15000e^{-x/5})(12000 - 12000e^{-y/8})}{10000}.$$

You would like to minimize your costs $C(x,y) = 30000x + 50000y$ subject to the constraint $S(x,y) = 14000$. Recall from Calculus III that we can use a Lagrange multiplier to perform this constrained minimization, namely, by solving the system of equations:

$$\begin{aligned} \nabla(C(x,y) - \lambda S(x,y)) &= \vec{0} \\ S(x,y) &= 14000 \end{aligned}$$

where ∇ denotes the gradient with respect to x and y. Write out this system of three equations for x, y, and λ, and solve them via Newton's Method. (If you want an intuitive feeling for what λ represents, observe that $\frac{\partial C}{\partial x} = \lambda \frac{\partial S}{\partial x}$, so it represents a proportionality factor between the marginal change in cost and the marginal change in sales obtained by adding an additional salesperson).

Multivariable Newton's Method Choice 3: Fractal Basins of Attraction

A system of nonlinear equations often has more than one solution. For each solution, there is a set of initial guesses called the *basin of attraction* that will converge to that solution under Newton's method. This basin of attraction can have an appealing structure, such as a fractal.

Consider the equations:

$$\begin{aligned} x^3 - 3xy^2 &= 1 \\ 3x^2y - y^3 &= 0 \end{aligned}$$

(These are the real and imaginary parts of $z^3 = 1$). The only solutions to this system of equations are $(1, 0)$, $(-1/2, \sqrt{3}/2)$, and $(-1/2, -\sqrt{3}/2)$.

(a) Set up a Mathematica function `nm[x_,y_]` that will run Newton's method with initial conditions x,y and return the list {1,0,0} if it converges to $(1, 0)$ in 15 steps, {0,1,0} if it coverges to $(-1/2, \sqrt{3}/2)$ in 15 steps, {0,0,1} if it coverges to $(-1/2, -\sqrt{3}/2)$ in 15 steps, and {0,0,0} if it does not converge in 15 steps. Test this function with three initial guesses, one close to each of the three solutions, to show that Newton's Method converges to those solutions.

(b) Use the function `nm` to generate a color picture of the basins of attraction in the square $-1 \le x, y \le 1$, as follows. At the beginning of the code, initialize `pic` as an empty list using the command `pic=` {} Then, write a `Do` loop that calls `nm` on a grid of points. After each computation of `nm`, create a rectangle using the command

```
pic = Append[pic,Graphics[{RGBColor[r,g,b],Rectangle[{x1,y1},{x2,y2}]}]]
```

where r, g, and b are the values output by `nm` (feel free to change the color scheme; the values I suggested will make a picture that is red, green, blue, and black). After the entire loop is finished, show the picture using the command

```
Show[pic]
```

WARNING: Be careful not to run too fine a grid. The computations can be long and may fill up the memory of your computer. Save your Mathematica notebook before you create the picture, in case Mathematica crahses. Start slowly (say, a 10-by-10 grid) and work up gradually until the computation becomes too intensive.

(c) Choose a region that looks interesting and zoom in to see the finer structure. Feel free to use multiple computers to do various runs.