Math 222 Fall 2002—HW #5 Solutions

Problem 1. Consider the IVP $x' = f(t,x), \quad x(t_0) = x_0$. In class we derived several algorithms for approximating $x(t_0+h)$, where h is a small time-step. One of these algorithms was the "2nd order Taylor" algorithm:

$$x(t_0+h) \approx x_0 + hf(t_0,x_0) + \frac{h^2}{2}\left(\frac{\partial f}{\partial t}(t_0,x_0) + f(t_0,x_0)\frac{\partial f}{\partial x}(t_0,x_0)\right).$$

In a similar way, derive the following "3rd order Taylor" algorithm for solving an IVP:

$$\begin{aligned} x(t_0+h) \approx\ & x_0 + hf(t_0,x_0) + \frac{h^2}{2}\left(\frac{\partial f}{\partial t}(t_0,x_0) + f(t_0,x_0)\frac{\partial f}{\partial x}(t_0,x_0)\right) \\ & + \frac{h^3}{6}\left[\frac{\partial^2 f}{\partial t^2}(t_0,x_0) + 2f(t_0,x_0)\frac{\partial^2 f}{\partial t \partial x}(t_0,x_0) + \frac{\partial f}{\partial t}(t_0,x_0)\frac{\partial f}{\partial x}(t_0,x_0)\right. \\ & \left. + \left(\frac{\partial f}{\partial x}(t_0,x_0)\right)^2 f(t_0,x_0) + (f(t_0,x_0))^2\frac{\partial^2 f}{\partial x^2}(t_0,x_0)\right] \end{aligned}$$

We know that the Taylor series expansion for $x(t_0+h)$ is

$$x(t_0+h) = x(t_0) + hx'(t_0) + \frac{h^2}{2!}x''(t_0) + \frac{h^3}{6}x'''(t_0) + \cdots$$

We know that $x(t_0) = x_0$ and we know from the differential equation that $x'(t_0) = f(t_0,x_0)$. In class, we showed that:

$$x''(t) = \frac{d}{dt}f(t,x(t)) = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x}\frac{dx}{dt} = \frac{\partial f}{\partial t} + f\frac{\partial f}{\partial x}.$$

So,

$$\begin{aligned} x'''(t) &= \frac{d}{dt}\left[\frac{\partial f}{\partial t}(t,x(t)) + f(t,x(t))\frac{\partial f}{\partial x}(t,x(t))\right] \\ &= \frac{\partial^2 f}{\partial t^2} + \frac{\partial^2 f}{\partial x \partial t}\frac{dx}{dt} + \frac{\partial f}{\partial t}\frac{\partial f}{\partial x} + \frac{\partial f}{\partial x}\frac{dx}{dt}\frac{\partial f}{\partial x} + f\frac{\partial^2 f}{\partial t \partial x} + f\frac{\partial^2 f}{\partial x^2}\frac{dx}{dt} \\ &= \frac{\partial^2 f}{\partial t^2} + \frac{\partial^2 f}{\partial x \partial t}\cdot f + \frac{\partial f}{\partial t}\frac{\partial f}{\partial x} + \frac{\partial f}{\partial x}\cdot f \cdot \frac{\partial f}{\partial x} + f\frac{\partial^2 f}{\partial t \partial x} + f\frac{\partial^2 f}{\partial x^2}\cdot f \\ &= \frac{\partial^2 f}{\partial t^2} + 2f\frac{\partial^2 f}{\partial x \partial t} + \frac{\partial f}{\partial t}\frac{\partial f}{\partial x} + f\left(\frac{\partial f}{\partial x}\right)^2 + (f)^2\frac{\partial^2 f}{\partial x^2} \end{aligned}$$

So, the 3rd order Taylor algorithm is:

$$x_{n+1} = x_n + hf + \frac{h^2}{2}\left[\frac{\partial f}{\partial t} + f\frac{\partial f}{\partial x}\right] + \frac{h^3}{6}\left[\frac{\partial^2 f}{\partial t^2} + 2f\frac{\partial^2 f}{\partial x \partial t} + \frac{\partial f}{\partial t}\frac{\partial f}{\partial x} + f\left(\frac{\partial f}{\partial x}\right)^2 + (f)^2\frac{\partial^2 f}{\partial x^2}\right],$$

where f and all its derivatives are evaluated at (t_n, x_n).

Problem 2. Consider the Runge-Kutta-2 method we saw in class:

$$x(t_0+h) \approx x_0 + hf\left(t_0 + \frac{h}{2}, x_0 + \frac{h}{2}f(t_0,x_0)\right)$$

and an additional RK2 method:

$$x(t_0+h) \approx x_0 + \frac{1}{4}hf(t_0,x_0) + \frac{3}{4}hf\left(t_0+\frac{2h}{3}, x_0+\frac{2h}{3}f(t_0,x_0)\right)$$

(a) Determine which method should be expected to be more accurate, by Taylor-expanding through the h^3 term and comparing to the 3rd order Taylor expansion from Problem 1. How should their accuracy compare to the 2nd order Taylor method (which has no h^3 term)?

We know that in general,

$$f(t+\Delta t, x+\Delta x) = f(t,x) + \frac{\partial f}{\partial t}\Delta t + \frac{\partial f}{\partial x}\Delta x + \frac{1}{2}\frac{\partial^2 f}{\partial t^2}(\Delta t)^2 + \frac{\partial^2 f}{\partial t \partial x}\Delta t \Delta x + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(\Delta x)^2 + \cdots$$

We apply this to the RK2 formula from class, which includes the term

$$f\left(t_0+\frac{h}{2}, x_0+\frac{h}{2}f(t_0,x_0)\right),$$

which matches our formula above with $\Delta t = \frac{h}{2}$ and $\Delta x = \frac{h}{2}f(t_0,x_0)$. So,

$$\begin{aligned} f\left(t_0+\frac{h}{2}, x_0+\frac{h}{2}f\right) &= f(t_0,x_0) + \frac{\partial f}{\partial t}\frac{h}{2} + \frac{\partial f}{\partial x}\frac{h}{2}f + \frac{1}{2}\frac{\partial^2 f}{\partial t^2}(\frac{h}{2})^2 + \frac{\partial^2 f}{\partial t \partial x}\frac{h}{2}\frac{h}{2}f + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(\frac{h}{2}f)^2 + \cdots \\ &= f + \frac{h}{2}\frac{\partial f}{\partial t} + \frac{h}{2}f\frac{\partial f}{\partial x} + \frac{h^2}{8}\frac{\partial^2 f}{\partial t^2} + \frac{h^2}{4}f\frac{\partial^2 f}{\partial t \partial x} + \frac{h^2}{8}(f)^2\frac{\partial^2 f}{\partial x^2} + \cdots \end{aligned}$$

(f and its derivatives always evaluated at (t_0, x_0)). Inserted into the RK2 algorithm:

$$x_1 = x_0 + hf\left(t_0+\frac{h}{2}, x_0+\frac{h}{2}f(t_0,x_0)\right)$$

we find:

$$\begin{aligned} x_1 &= x_0 + h\left[f + \frac{h}{2}\frac{\partial f}{\partial t} + \frac{h}{2}f\frac{\partial f}{\partial x} + \frac{h^2}{8}\frac{\partial^2 f}{\partial t^2} + \frac{h^2}{4}f\frac{\partial^2 f}{\partial t \partial x} + \frac{h^2}{8}(f)^2\frac{\partial^2 f}{\partial x^2} + \cdots\right] \\ &= x_0 + hf + \frac{h^2}{2}\frac{\partial f}{\partial t} + \frac{h^2}{2}f\frac{\partial f}{\partial x} + \frac{h^3}{8}\frac{\partial^2 f}{\partial t^2} + \frac{h^3}{4}f\frac{\partial^2 f}{\partial t \partial x} + \frac{h^3}{8}(f)^2\frac{\partial^2 f}{\partial x^2} + \cdots \end{aligned}$$

Similarly, we apply the Taylor expansion to the new RK2 formula, which includes the term

$$f\left(t_0+\frac{2h}{3}, x_0+\frac{2h}{3}f(t_0,x_0)\right),$$

which matches our formula above with $\Delta t = \frac{2h}{3}$ and $\Delta x = \frac{2h}{3}f(t_0,x_0)$. So,

$$\begin{aligned} f\left(t_0+\frac{2h}{3}, x_0+\frac{2h}{3}f\right) &= f(t_0,x_0) + \frac{\partial f}{\partial t}\frac{2h}{3} + \frac{\partial f}{\partial x}\frac{2h}{3}f + \frac{1}{2}\frac{\partial^2 f}{\partial t^2}(\frac{2h}{3})^2 + \frac{\partial^2 f}{\partial t \partial x}\frac{2h}{3}\frac{2h}{3}f + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(\frac{2h}{3}f)^2 + \cdots \\ &= f + \frac{2h}{3}\frac{\partial f}{\partial t} + \frac{2h}{3}f\frac{\partial f}{\partial x} + \frac{2h^2}{9}\frac{\partial^2 f}{\partial t^2} + \frac{4h^2}{9}f\frac{\partial^2 f}{\partial t \partial x} + \frac{2h^2}{9}(f)^2\frac{\partial^2 f}{\partial x^2} + \cdots \end{aligned}$$

(f and its derivatives always evaluated at (t_0, x_0)). Inserted into the RK2 algorithm:

$$x_1 = x_0 + \frac{1}{4}hf(t_0, x_0) + \frac{3}{4}hf\left(t_0 + \frac{h}{2}, x_0 + \frac{h}{2}f(t_0, x_0)\right)$$

we find:

$$\begin{aligned} x_1 &= x_0 + \frac{1}{4}hf(t_0,x_0) + \frac{3}{4}h\left[f + \frac{2h}{3}\frac{\partial f}{\partial t} + \frac{2h}{3}f\frac{\partial f}{\partial x} + \frac{2h^2}{9}\frac{\partial^2 f}{\partial t^2} + \frac{4h^2}{9}f\frac{\partial^2 f}{\partial t \partial x} + \frac{2h^2}{9}(f)^2\frac{\partial^2 f}{\partial x^2} + \cdots\right] \\ &= x_0 + hf(t_0,x_0) + \frac{h^2}{2}\frac{\partial f}{\partial t} + \frac{h^2}{2}f\frac{\partial f}{\partial x} + \frac{h^3}{6}\frac{\partial^2 f}{\partial t^2} + \frac{h^3}{3}f\frac{\partial^2 f}{\partial t \partial x} + \frac{h^3}{6}(f)^2\frac{\partial^2 f}{\partial x^2} + \cdots \end{aligned}$$

If we compare both RK2 methods to the 3rd order Taylor formula in Problem 1, we find that they agree exactly through the h^2 terms, and the three h^3 terms that appear in the RK2 expansions appear in the 3rd order Taylor formula, though in the first RK2 expansion, they have coefficients $\frac{1}{8}$,$\frac{1}{4}$, and $\frac{1}{8}$ whereas in the 3rd order Taylor formula they have coefficients $\frac{1}{6}$, $\frac{1}{3}$, and $\frac{1}{6}$ (the second RK2 expansion gets these coefficients correct). In contrast, the 2nd order Taylor formula agrees with the 3rd order Taylor formula through the h^2 terms, but it has no h^3 terms. So, the second RK2 algorithm is the best approximation, following by the first RK2 algorithm and then T2 (it is better to be like the first RK2 expansion, and have some of the T3 h^3 terms with slightly wrong coefficients, but of the correct sign and approximate size at least, than to be like T2 and be missing them altogether).

(b) Perform one step of 2nd order Taylor and each of the RK2 algorithms on the IVP

$$x' = (1+t)(1+x^2), \quad x(0) = 1$$

with $h = 0.2$, and compare their accuracy to the exact solution $x(t) = \tan\left(\frac{\pi+4t+2t^2}{4}\right)$.

We have $x' = f(t, x)$ with $f(t, x) = (1 + t)(1 + x^2)$. To do the 2nd order Taylor algorithm, we need to compute:

$$\frac{\partial f}{\partial t} = 1 + x^2, \quad \frac{\partial f}{\partial x} = 2x(1+t)$$

Now, the 2nd order Taylor algorithm says

$$x_1 = x_0 + hf + \frac{h^2}{2}\left[\frac{\partial f}{\partial t} + f\frac{\partial f}{\partial x}\right],$$

where f and all its derivatives are evaluated at $(t_0, x_0) = (0, 1)$. Since $f(0, 1) = 2$, $\frac{\partial f}{\partial t}(0, 1) = 2$, and $\frac{\partial f}{\partial x}(0, 1) = 2$, we have:

$$x_1 = 1 + (0.2)(2) + \frac{(0.2)^2}{2}[2 + (2)(2)] = 1 + 0.4 + 0.02(6) = 1.52$$

The first Runge-Kutta-2 algorithm says

$$\begin{aligned} x_1 &= x_0 + hf\left(t_0 + \frac{h}{2}, x_0 + \frac{h}{2}f(t_0,x_0)\right) \\ &= 1 + 0.2f\left(0 + \frac{0.2}{2}, 1 + \frac{0.2}{2}(2)\right) \\ &= 1 + 0.2f(0.1, 1.2) = 1 + (0.2)(1.1)(1 + (1.2)^2) = 1.5368. \end{aligned}$$

The second Runge-Kutta-2 algorithm says

$$\begin{aligned} x_1 &= x_0 + \frac{1}{4}hf(t_0, x_0) + \frac{3}{4}hf\left(t_0 + \frac{2h}{3}, x_0 + \frac{2h}{3}f(t_0, x_0)\right) \\ &= 1 + \frac{1}{4}0.2(2) + \frac{3}{4}0.2f\left(0 + \frac{2(0.2)}{3}, 1 + \frac{2(0.2)}{3}(2)\right) \\ &= 1 + 0.1 + 0.15f(2/15, 19/15) = 1 + 0.1 + (0.15)(17/15)(1 + (19/15)^2) = 1.542756. \end{aligned}$$

The exact solution has $x(0.2) = \tan\left(\frac{\pi+0.88}{4}\right) = 1.576056$. Sure enough, the second RK2 solution is most accurate, followed by the first RK2, and then T2.

3. We have seen some methods earlier in the course for approximating definite integrals $\int_a^b f(x)dx$. Show how you can use an IVP to approximate the value of the integral $\int_0^3 e^{-x^2/2}dx$.

If we define $g(t) = \int_0^t e^{-x^2/2}dx$, then we know by the Fundamental Theorem of Calculus that $g'(t) = e^{-t^2/2}$. What we are after is $g(3) = \int_0^3 e^{-x^2/2}dx$, and we know that $g(0) = \int_0^0 e^{-x^2/2}dx = 0$. Thus, we have a standard initial value problem:

$$\frac{dg}{dt} = e^{-t^2/2}, \quad g(0) = 0$$

to solve. We can solve it by the RK4 algorithm with $h = 0.1$, as shown in the "HW5 Solutions" notebook. The final value for the integral is approximately 1.249930438. We know that in general, RK4 should give error $O((t_f - t_0)h^4)$, which means approximately $3(0.1)^4 = .0003$.

Problem 4. Use a Runge-Kutta-4 algorithm with $h = 0.5$ to solve

$$x' = x(x-1)(x-2)/4, \quad x(0) = 0.5;$$

compare to the exact solution

$$x(t) = \frac{-0.25 - 0.75e^{t/2} + \sqrt{0.0625 + 0.1875e^{t/2}}}{-0.25 - 0.75e^{t/2}}$$

at $t = 1$. Use a Runge-Kutta-4 algorithm with $h = 0.5$ to solve

$$x' = x(x-1)(x-2)/4, \quad x(0) = 2.5;$$

compare to the exact solution

$$x(t) = \frac{-2.25 + 1.25e^{t/2} - \sqrt{5.0625 - 2.8125e^{t/2}}}{-2.25 + 1.25e^{t/2}}$$

at $t = 1$. Compare the accuracy of your two runs. On paper, analyze the instability of the IVP to explain this difference.

The computer runs are shown in the "HW5 Solutions" notebook. The error in the first case is less than 10^{-7}, while in the second case, it is around 0.08. We can compute, given the right-hand side $f(x) = x(x-1)(x-2)/4 = (x^3 - 3x^2 + 2x)/4$, that

$$\frac{\partial f}{\partial x} = (3x^2 - 6x + 2)/4$$

Now, for the first run, x goes from 0.5 to a final value of 0.59, and in that range $\frac{\partial f}{\partial x}$ is between -0.06 and -0.13, so we expect stability (and hence the high accuracy of the result). On the other hand, for the second run, x goes from 2.5 to a final value of 4.4, and in that range $\frac{\partial f}{\partial x}$ is between 1.4 and 8.4, so we expect instability (and hence the lower accuracy of the result).