Math 222 Fall 2002—HW #5 Due: Fri. Dec. 6

Reading (from Cheney and Kincaid): §8 (pp. 370–374), 8.1 (pp. 374–380), 8.2 (pp. 383-389), 8.3 (pp. 395–398, and "Stability Analysis" section), 9.1 (just the sections "Vector Notation", "Systems of ODEs", and "Runge-Kutta Method"), 9.2 (through the section "Higher-Order Differential Equations").

1. Consider the IVP $x' = f(t,x), \quad x(t_0) = x_0$. In class we derived several algorithms for approximating $x(t_0+h)$, where h is a small time-step. One of these algorithms was the "2nd order Taylor" algorithm:

$$x(t_0+h) \approx x_0 + hf(t_0,x_0) + \frac{h^2}{2}\left(\frac{\partial f}{\partial t}(t_0,x_0) + f(t_0,x_0)\frac{\partial f}{\partial x}(t_0,x_0)\right)$$

which we derived by expanding $x(t_0+h)$ in a Taylor series up to the h^2 term, and then applying the differential equation to write derivatives of $x(t)$ in terms of f and its derivatives. In a similar way, derive the following "3rd order Taylor" algorithm for solving an IVP:

$$\begin{aligned} x(t_0+h) \quad &\approx \quad x_0 + hf(t_0,x_0) + \frac{h^2}{2}\left(\frac{\partial f}{\partial t}(t_0,x_0) + f(t_0,x_0)\frac{\partial f}{\partial x}(t_0,x_0)\right) \\ &+\frac{h^3}{6}\left[\frac{\partial^2 f}{\partial t^2}(t_0,x_0) + 2f(t_0,x_0)\frac{\partial^2 f}{\partial t\partial x}(t_0,x_0) + \frac{\partial f}{\partial t}(t_0,x_0)\frac{\partial f}{\partial x}(t_0,x_0)\right. \\ &\left. + \left(\frac{\partial f}{\partial x}(t_0,x_0)\right)^2 f(t_0,x_0) + (f(t_0,x_0))^2\frac{\partial^2 f}{\partial x^2}(t_0,x_0)\right] \end{aligned}$$

2. Consider the Runge-Kutta-2 method we saw in class:

$$x(t_0+h) \approx x_0 + hf\left(t_0 + \frac{h}{2}, x_0 + \frac{h}{2}f(t_0,x_0)\right)$$

and an additional RK2 method:

$$x(t_0+h) \approx x_0 + \frac{1}{4}hf(t_0,x_0) + \frac{3}{4}hf\left(t_0 + \frac{2h}{3}, x_0 + \frac{2h}{3}f(t_0,x_0)\right)$$

(a) Determine which method should be expected to be more accurate, by Taylor-expanding through the h^3 term and comparing to the 3rd order Taylor expansion from Problem 1. How should their accuracy compare to the 2nd order Taylor method (which has no h^3 term)? Here is the relevant formula for 2-variable Taylor expansions:

$$\begin{aligned} f(t+\Delta t, x+\Delta x) \quad &\approx \quad f(t,x) + \frac{\partial f}{\partial t}(t,x)\Delta t + \frac{\partial f}{\partial x}(t,x)\Delta x \\ &+\frac{1}{2}\frac{\partial^2 f}{\partial t^2}(t,x)(\Delta t)^2 + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t,x)(\Delta x)^2 + \frac{\partial^2 f}{\partial t\partial x}(t,x)(\Delta t)(\Delta x) \end{aligned}$$

(b) Perform one step of 2nd order Taylor and each of the RK2 algorithms on the IVP

$$x' = (1+t)(1+x^2), \quad x(0) = 1$$

with $h = 0.2$, and compare their accuracy to the exact solution $x(t) = \tan\left(\frac{\pi+4t+2t^2}{4}\right)$.

The following two problems require computations using Runge-Kutta-4 algorithms. On the course Web page, I've provided sample RK4 Mathematica code that you can alter to do the computations.

3. We have seen some methods earlier in the course for approximating definite integrals $\int_a^b f(x)dx$. Show how you can use an IVP to approximate the value of the integral $\int_0^3 e^{-x^2/2}dx$. (Hint: Consider the following function of t: $g(t) = \int_0^t e^{-x^2/2}dx$.) Use a Runge-Kutta-4 algorithm to determine an approximate value for this integral, and give an estimate for the accuracy of your answer, given what you know about the accuracy of RK4.

4. Use a Runge-Kutta-4 algorithm with $h = 0.5$ to solve

$$x' = x(x-1)(x-2)/4, \quad x(0) = 0.5;$$

compare to the exact solution

$$x(t) = \frac{-0.25 - 0.75e^{t/2} + \sqrt{0.0625 + 0.1875e^{t/2}}}{-0.25 - 0.75e^{t/2}}$$

at $t = 1$. Use a Runge-Kutta-4 algorithm with $h = 0.5$ to solve

$$x' = x(x-1)(x-2)/4, \quad x(0) = 2.5;$$

compare to the exact solution

$$x(t) = \frac{-2.25 + 1.25e^{t/2} - \sqrt{5.0625 - 2.8125e^{t/2}}}{-2.25 + 1.25e^{t/2}}$$

at $t = 1$. Compare the accuracy of your two runs. On paper, analyze the instability of the IVP to explain this difference.