Math 222 Fall 2002—HW #4 Solutions

1. (a) Recall our pseudo-random-number generator from class:

$$x_{n+1} = \textbf{fractional part of } (\pi + x_n)^5.$$

For simplicity, assume that our computer stores numbers exactly (i.e., keeps an infinite number of decimal places). There is a seed $x_0 \in (0,1)$ so that $x_0 = x_1 = x_2 = \ldots$ (in fact there are many such seeds). Determine one such x_0 to 8 decimal places.

We seek $x_0 \in (0,1)$ so that the fractional part of $(\pi + x_0)^5$ equals x_0. Observe that as x_0 ranges from 0 to 1, the quantity $(\pi + x_0)^5$ ranges from approximately 306.02 to 1218.53. So, for each integer n between 307 and 1218, there should be a solution z to

$$z + n = (\pi + z)^5$$

and this z will be equal to the fractional part of $(\pi + z)^5$ as we want. So, we randomly choose $n = 500$ and solve

$$z + 500 = (\pi + z)^5$$

by Newton's Method. We let

$$f(z) = (\pi + z)^5 - z - 500$$

so that

$$f'(z) = 5(\pi + z)^4 - 1$$

and make an initial guess $z_0 = 0.5$ and then compute:

$$z_1 = z_0 - \frac{f(z_0)}{f'(z_0)} = 0.3407051951$$

and then

$$z_2 = z_1 - \frac{f(z_1)}{f'(z_1)} = 0.3247302351$$

and then

$$z_3 = z_2 - \frac{f(z_2)}{f'(z_2)} = 0.324581422$$

and then

$$z_4 = z_3 - \frac{f(z_3)}{f'(z_3)} = 0.324581409$$

and then

$$z_5 = z_4 - \frac{f(z_4)}{f'(z_4)} = 0.324581409$$

So, to 8 decimal places,

$$x_0 = 0.32458141$$

is one such seed.

(b) Increase your x_0 from part (a) by 10^{-7} and do 2 steps of the pseudo-random-number generator, using as many digits as your calculator reports to you.

I start the pseudo-random number generator with $x_0 = 0.32458142$ and then compute x_1 to be the fractional part of $(\pi + x_0)^5$, which is 0.324589167, and then compute x_2 to be the fractional part of $(\pi + x_1)^5$, which is 0.330180392.

(c) What is the significance of part (a)? Of part (b)?

Part (a) shows us that on a "perfect" infinite-precision computer, this pseudo-random number generator can exhibit some very non-random behaviors, i.e., repeating the same number forever. However, part (b) reassures us that these repeating seeds are very unstable, in the sense that if you start quite close to them (but not right on them), the pseudo-random numbers diverge fairly rapidly from them. So, on a finite-precision real computer, we will probably never be able to start right on the fixed seeds, so, though we should still be cautious about how random the pseudo-random-number generator really is, we will probably at least not observe the really non-random behavior of a fixed point.

2. Let T be the triangle with vertices $(0, 0)$, $(1, 0)$, and $(0, 1)$.

(a) Assuming you have a function `Random[ ]` that selects a random number uniformly from $[0, 1]$, how can you select a random point (x, y) uniformly from T? ("uniformly from T" means that the probability of landing in some subset of T depends only on the area of that subset, not on its location).

Let x be chosen by calling `Random[ ]`, and y chosen by calling `Random[ ]` again. If $x + y > 1$, i.e. (x,y) is outside of T, then reject (x, y) and start over by choosing new x and y as above. If $x + y \leq 1$, then accept (x, y) as your random point in T.

(b) How could you use `Random[ ]` to approximate the area of T?

Do the following N times: let x be chosen by calling `Random[ ]`, and y chosen by calling `Random[ ]` again. Let M be the number of these random (x, y) that have $x + y \leq 1$. Then M/N is an approximation for the area of T.

3. Let $f(x) = \frac{1}{\pi(1+x^2)}$

(a) Show that f is a probability density. Sketch the graph of f.

We compute:

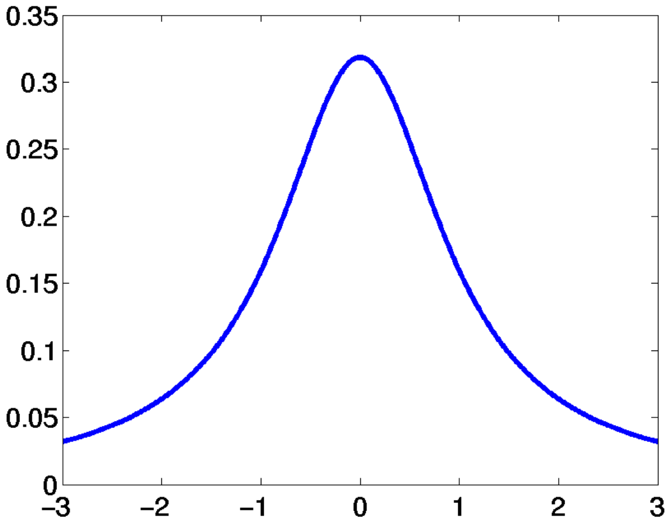
$$\int_{-\infty}^{\infty} f(x)dx = \frac{\arctan(x)}{\pi}\bigg|_{-\infty}^{\infty}$$

Since $\lim_{x\to\infty} \arctan(x) = \pi/2$ and $\lim_{x\to-\infty} \arctan(x) = -\pi/2$, we have

$$\int_{-\infty}^{\infty} f(x)dx = \frac{\pi/2}{\pi} - \frac{-\pi/2}{\pi} = 1$$

and thus f is a probability density.

The graph of f is shown below.



(b) Using `Random[ ]`, described in Problem 2, describe in detail an algorithm for computing a random number with probability density f.

We compute:

$$F(x) = \int_{-\infty}^{x} f(y)dy = \left.\frac{\arctan(y)}{\pi}\right|_{-\infty}^{x} = \frac{\arctan(x)}{\pi} - \frac{-\pi/2}{\pi} = \frac{\arctan(x)}{\pi} + \frac{1}{2}$$

We need to compute $F^{-1}(y)$, so we write $y = F(x)$, or

$$y = \frac{\arctan(x)}{\pi} + \frac{1}{2}$$

and solve for x:

$$\begin{aligned} y - \frac{1}{2} &= \frac{\arctan(x)}{\pi} \\ \pi y - \frac{\pi}{2} &= \arctan(x) \\ \tan\left(\pi y - \frac{\pi}{2}\right) &= x \end{aligned}$$

So, if we let y be chosen at random using `Random[ ]` and define $x = \tan\left(\pi y - \frac{\pi}{2}\right)$, then x will be a random number with density f.

4. In class we claimed that if you estimate the area under the curve $y = f(x)$ between $x = a$ and $x = b$ with N rectangles with heights $f(x_j)$ for x_j the midpoint of the jth subinterval, then the error in area per rectangle is $O(1/N^3)$. You will verify that here.

Let c be the midpoint of one of the rectangles. Then the left endpoint of the rectangle is at $x = c - \frac{b-a}{2N}$ and the right endpoint is at $x = c + \frac{b-a}{2N}$. What is the area of the rectangle? Use the Taylor series expansion of $f(x)$ about $x = c$ to determine

a series expansion for

$$\int_{c-\frac{b-a}{2N}}^{c+\frac{b-a}{2N}} f(x)dx$$

and show that the first term that differs from the rectangle area is a term of order $1/N^3$.

The area of the rectangle is $f(c) \cdot \frac{b-a}{N}$ because its height is $f(c)$ and its width is $\left(c + \frac{b-a}{2N}\right) - \left(c - \frac{b-a}{2N}\right) = \frac{b-a}{N}$.

Now, we plug the series expansion of $f(x)$ about c:

$$f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots$$

into the given integral:

$$A_{exact} = \int_{c-\frac{b-a}{2N}}^{c+\frac{b-a}{2N}} f(x)dx = \int_{c-\frac{b-a}{2N}}^{c+\frac{b-a}{2N}} \left[f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots \right] dx$$

Before we evaluate this, we make simple change of variables $u = x - c$ (so that $du = dx$):

$$A_{exact} = \int_{-\frac{b-a}{2N}}^{\frac{b-a}{2N}} \left[f(c) + f'(c)u + \frac{f''(c)}{2!}u^2 + \cdots \right] du$$

We note that the u term is an odd function, so its integral will vanish over the given range. Thus,

$$A_{exact} = \int_{-\frac{b-a}{2N}}^{\frac{b-a}{2N}} \left[f(c) + \frac{f''(c)}{2!}u^2 + \cdots \right] du$$

By a similar argument, the remaining two terms in the brackets are even, so their integral equals twice the integral from 0 to $\frac{b-a}{2N}$:

$$A_{exact} = 2\int_{0}^{\frac{b-a}{2N}} \left[f(c) + \frac{f''(c)}{2!}u^2 + \cdots \right] du$$

OK, now we have to stop stalling and evaluate the integral:

$$\begin{aligned} A_{exact} &= 2\left[f(c)u + \frac{f''(c)}{2!}\frac{u^3}{3} + \cdots \right]_0^{\frac{b-a}{2N}} \\ &= 2\left[f(c)\left(\frac{b-a}{2N}\right) + \frac{f''(c)}{6}\left(\frac{b-a}{2N}\right)^3 + \cdots \right] \\ &= f(c) \cdot \frac{b-a}{N} + \frac{f''(c)}{24} \cdot \frac{(b-a)^3}{N^3} + \cdots \end{aligned}$$

Note that the first term of this expansion agrees with the rectangle area, as we expected, and the next nonzero term is a multiple of $1/N^3$, as we wanted to prove.