Math 222 Fall 2002—HW #4 Due: Fri. Oct. 25

Reading (from Cheney and Kincaid):

§11.1–11.3

Problems:

1. (a) Recall our pseudo-random-number generator from class:

$$x_{n+1} = \text{fractional part of } \ (\pi + x_n)^5.$$

For simplicity, assume that our computer stores numbers exactly (i.e., keeps an infinite number of decimal places). There is a seed $x_0 \in (0,1)$ so that $x_0 = x_1 = x_2 = \ldots$ (in fact there are many such seeds). Determine one such x_0 to 8 decimal places.

(b) Increase your x_0 from part (a) by 10^{-7} and do 2 steps of the pseudo-random-number generator, using as many digits as your calculator reports to you.

(c) What is the significance of part (a)? Of part (b)?

2. Let T be the triangle with vertices $(0,0)$, $(1,0)$, and $(0,1)$.

(a) Assuming you have a function `Random[ ]` that selects a random number uniformly from $[0,1]$, how can you select a random point (x,y) uniformly from T? ("uniformly from T" means that the probability of landing in some subset of T depends only on the area of that subset, not on its location).

(b) How could you use `Random[ ]` to approximate the area of T?

3. Let $f(x) = \frac{1}{\pi(1+x^2)}$

(a) Show that f is a probability density. Sketch the graph of f.

(b) Using `Random[ ]`, described in Problem 2, describe in detail an algorithm for computing a random number with probability density f.

4. In class we claimed that if you estimate the area under the curve $y = f(x)$ between $x = a$ and $x = b$ with N rectangles with heights $f(x_j)$ for x_j the midpoint of the jth subinterval, then the error in area per rectangle is $O(1/N^3)$. You will verify that here.

Let c be the midpoint of one of the rectangles. Then the left endpoint of the rectangle is at $x = c - \frac{b-a}{2N}$ and the right endpoint is at $x = c + \frac{b-a}{2N}$. What is the area of the rectangle? Use the Taylor series expansion of $f(x)$ about $x = c$ to determine a series expansion for

$$\int_{c-\frac{b-a}{2N}}^{c+\frac{b-a}{2N}} f(x)dx$$

and show that the first term that differs from the rectangle area is a term of order $1/N^3$.