Math 222 Fall 2002—HW #2 Solutions

1. Consider the system of 2 linear equations:

$$\begin{aligned} 10^{-5}x + y &= 3 \\ x - \frac{1}{3}y &= -\frac{1}{3} \end{aligned}$$

Compute the exact solution. Compute the solution as computed on the `LUCKY-7` computer by naive Gaussian elimination. Compute the solution as computed on the `LUCKY-7` computer by naive Gaussian elimination if the order of the two equations is first swapped. Compare the accuracy of the two approximate solutions.

In exact arithmetic, we know that $y = 3 - 10^{-5}x$, so that

$$x - \frac{3 - 10^{-5}x}{3} = -\frac{1}{3},$$

which implies

$$\left(1 - \frac{1}{300000}\right)x = \frac{2}{3},$$

or

$$\frac{299999}{300000}x = \frac{2}{3},$$

or

$$x = \frac{200000}{299999}.$$

In decimal notation,

$$x = 0.66666888889629\ldots.$$

Next, we solve for y exactly:

$$y = 3 - 10^{-5}x = 3 - \frac{2}{299999}.$$

In decimal notation,

$$y = 2.999993333311111037\ldots.$$

Now we'll do Gaussian elimination in the `LUCKY-7` system. First, let's rewrite the system in `LUCKY-7` form, keeping in mind that we can only store 7 digits starting from the first nonzero digit:

$$\begin{aligned} 10^{-5}x + y &= 3 \\ x - 0.3333333y &= -0.3333333 \end{aligned}$$

First, we take -10^5 times the first equation and add it to the second equation, replacing the second equation:

$$\begin{aligned} 10^{-5}x + y &= 3 \\ (-100000 - 0.3333333)\,y &= (-300000 - 0.3333333) \end{aligned}$$

Now, in `LUCKY-7`, we can only store 7 digits starting from the first nonzero digit, so these coefficients would be stored as:

$$\begin{aligned} 10^{-5}x + y &= 3 \\ -100000.3y &= -300000.3 \end{aligned}$$

Now we can solve for y. In exact arithmetic, $\frac{-300000.3}{-100000.3} = 2.999994000\ldots$, which in `LUCKY-7` would get stored as:

$$y = 2.999994$$

Then we solve for x:

$$10^{-5}x = 3 - 2.999994 = 0.000006,$$

which implies that

$$x = 0.6$$

Now we'll do Gaussian elimination in the `LUCKY-7` system with the order of the equations swapped. Let's rewrite the system with this new ordering:

$$\begin{aligned} x - 0.3333333y &= -0.3333333 \\ 10^{-5}x + y &= 3 \end{aligned}$$

First, we take -10^{-5} times the first equation and add it to the second equation, replacing the second equation:

$$\begin{aligned} x - 0.3333333y &= -0.3333333 \\ (1 + 0.000003333333)\, y &= 3 + 0.000003333333, \end{aligned}$$

Now, in `LUCKY-7`, we can only store 7 digits starting from the first nonzero digit, so these coefficients would be stored as:

$$\begin{aligned} x - 0.3333333y &= -0.3333333 \\ 1.000003y &= 3.000003, \end{aligned}$$

Now we can solve for y. In exact arithmetic, $\frac{-300000.3}{-100000.3} = 2.999994000\ldots$, which in `LUCKY-7` would get stored as:

$$y = 2.999994$$

Then we solve for x:

$$x - (0.3333333)(2.999994) = -0.3333333$$

In exact arithmetic, $(0.3333333)(2.999994) = 0.9999979000002$, which would get stored in `LUCKY-7` as:

$$x - 0.9999979 = -0.3333333$$

We would then find:

$$x = 0.9999979 - 0.3333333 = 0.6666646$$

Thus, in the first case, the errors in x and y are $0.066\ldots$ and $0.0000007\ldots$, respectively, whereas in the second case, we have errors of $0.0000042\ldots$ and $0.0000007\ldots$, respectively. Thus, there is much improved accuracy in x in the second case.

2. (§6.1, # 5): Consider

$$\mathbf{A} = \begin{bmatrix} 0.780 & 0.563 \\ 0.913 & 0.659 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 0.217 \\ 0.254 \end{bmatrix}$$

$$\tilde{\mathbf{x}} = \begin{bmatrix} 0.999 \\ -1.001 \end{bmatrix}, \quad \hat{\mathbf{x}} = \begin{bmatrix} 0.341 \\ -0.087 \end{bmatrix}$$

Compute residual vectors $\tilde{\mathbf{r}} = \mathbf{A}\tilde{\mathbf{x}} - \mathbf{b}$ and $\hat{\mathbf{r}} = \mathbf{A}\hat{\mathbf{x}} - \mathbf{b}$ and decide which of $\tilde{\mathbf{x}}$ or $\hat{\mathbf{x}}$ is the better solution vector. Now compute the error vectors $\tilde{\mathbf{e}} = \tilde{\mathbf{x}} - \mathbf{x}$ and $\hat{\mathbf{e}} = \hat{\mathbf{x}} - \mathbf{x}$, where $\mathbf{x} = [1, -1]^T$ is the exact solution. Discuss the implications of this example.

We find that:

$$\tilde{\mathbf{r}} = \begin{bmatrix} -0.001343 \\ -0.001572 \end{bmatrix}, \quad \hat{\mathbf{r}} = \begin{bmatrix} -0.000001 \\ 0 \end{bmatrix},$$

so that $\hat{\mathbf{x}}$ appears to be the more accurate solution, but then we observe that:

$$\tilde{\mathbf{e}} = \begin{bmatrix} -0.001 \\ -0.001 \end{bmatrix}, \quad \hat{\mathbf{e}} = \begin{bmatrix} -0.659 \\ 0.913 \end{bmatrix},$$

so that in fact, $\tilde{\mathbf{x}}$ is the more accurate solution. Thus, we can not be sure from an analysis of purely the residuals that one solution is, in fact, more accurate than another. Isn't that a little depressing?

3. (§3.2, # 1): Verify that when Newton's method is used to compute $\sqrt{R}$ by solving $x^2 = R$, the sequence of iterates is defined by

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{R}{x_n}\right).$$

We let $f(x) = x^2 - R$, so that we seek a solution to $f(x) = 0$. We compute that $f'(x) = 2x$. So, using the formula for Newton's method:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^2 + R}{2x_n} = x_n - \frac{x_n}{2} + \frac{R}{2x_n} = \frac{1}{2}\left(x_n + \frac{R}{x_n}\right).$$

4. Use the result of Problem 3 to compute the first 4 steps of Newton's method to solve $x^2 = 2$, starting at $x_0 = 1$. Compute the first 4 steps of the solution of $x^2 = 2$ by bisection, starting with $a = 1$, $b = 2$. Compute the first 4 terms of the Taylor series approximation to $\sqrt{2}$, using the result of Problem 1 on the last HW. Compare the accuracy of these three approximations.

We compute the iterates of Newton's method:

$$\begin{aligned}
x_0 &= 1 \\
x_1 &= \frac{1}{2}\left(x_0 + \frac{2}{x_0}\right) = \frac{1}{2}\left(1 + \frac{2}{1}\right) = 1.5 \\
x_2 &= \frac{1}{2}\left(x_1 + \frac{2}{x_1}\right) = \frac{1}{2}\left(1.5 + \frac{2}{1.5}\right) = 1.4166666 \\
x_3 &= \frac{1}{2}\left(x_2 + \frac{2}{x_2}\right) = \frac{1}{2}\left(1.4166666 + \frac{2}{1.4166666}\right) = 1.4142157 \\
x_4 &= \frac{1}{2}\left(x_3 + \frac{2}{x_3}\right) = \frac{1}{2}\left(1.4142157 + \frac{2}{1.4142157}\right) = 1.4142136
\end{aligned}$$

For bisection, $a_0 = 1$ and $b_0 = 2$, with $f(a_0) = 1-2 < 0$ and $f(b_0) = 4-2 > 0$. We have $c_0 = 1.5$ and $f(c_0) = 1.5^2 - 2 > 0$, so $a_1 = 1$ and $b_1 = 1.5$. We have $c_1 = 1.25$ and $f(c_1) = 1.25^2 - 2 < 0$, so $a_2 = 1.25$ and $b_2 = 1.5$. We have $c_2 = 1.375$ and $f(c_2) = 1.375^2 - 2 < 0$, so $a_3 = 1.375$ and $b_3 = 1.5$. Finally, we have $c_3 = 1.4375$ and $f(c_3) = 1.4375^2 - 2 > 0$, so $a_3 = 1.375$ and $b_3 = 1.4375$. Thus, the guess for the 4th step of bisection is $c_4 = 1.40625$.

Finally, we use the Taylor series:

$$(1+x)^{1/2} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \frac{5}{128}x^4 + \dots$$

to approximate:

$$2^{1/2} \approx 1 + \frac{1}{2} - \frac{1}{8} + \frac{1}{16} - \frac{5}{128} = 1.3984375.$$

Given that the exact value of $\sqrt{2}$ is $1.4142136\dots$, we see that Newton's method is accurate to at least 8 digits, bisection to 3 digits, and the Taylor series to 2 digits.

5. (§3.2, # 19): For what starting values will Newton's method converge if the function f is $f(x) = x^2/(1+x^2)$.

The only root of f is zero. Because of the W-shape of the graph, we presume that if the second iterate x_1 is closer to zero than the first iterate x_0, then all future iterates will get closer and closer to zero and thus Newton's method will converge, but otherwise Newton's method will diverge. Because of symmetry of f about zero, it is clear that we need only consider $x_0 > 0$, as the results will be symmetric about zero. Because of the W-shape, $x_1 < x_0$, so the only question is whether $x_1 > -x_0$.

Now, $f'(x) = \frac{(1+x^2)(2x)-x^2(2x)}{(1+x^2)^2} = \frac{2x}{(1+x^2)^2}$. So, $\frac{f(x)}{f'(x)} = \frac{x^2}{1+x^2} \cdot \frac{(1+x^2)^2}{2x} = \frac{x(1+x^2)}{2}$. Thus, Newton's iteration is:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n(1+x_n^2)}{2} = \frac{x_n(1-x_n^2)}{2}$$

Thus, $x_1 = \frac{x_0(1-x_0^2)}{2}$, and we want to know for which $x_0 > 0$ we have $x_1 > -x_0$, i.e.,

$$\frac{x_0(1-x_0^2)}{2} > -x_0,$$

or

$$x_0 + \frac{x_0(1-x_0^2)}{2} > 0,$$

or

$$\frac{x_0(3-x_0^2)}{2} > 0.$$

This is true if and only if $x_0 < \sqrt{3}$. Overall, Newton's method converges if $-\sqrt{3} < x_0 < \sqrt{3}$.