Math 222 Fall 2002—HW #1 Solutions

Problems (careful: don't do the "Computer Problems"):

1. (§1.2, # 1) Show that for $f(x) = (1+x)^n$, Eq. (6) takes the form

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \dots$$

For $n = 2, 3, 1/2$, give the particular form of this series. Use the last form to compute $\sqrt{1.0001}$ correct to 15 decimal places (rounded).

Since $f(x) = (1+x)^n$, we have that $f(0) = 1$. Since $f'(x) = n(1+x)^{n-1}$, we have that $f'(0) = n$. Since $f''(x) = n(n-1)(1+x)^{n-2}$, we have that $f''(0) = n(n-1)$. Since $f'''(x) = n(n-1)(n-2)(1+x)^{n-3}$, we have that $f'''(0) = n(n-1)(n-2)$. Inserting these derivatives into the Taylor series formula, we find the given Taylor series:

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \dots$$

For $n = 2$, we find:

$$(1+x)^2 = 1 + 2x + \frac{2(1)}{2!}x^2 + \frac{2(1)(0)}{3!}x^3 + \dots = 1 + 2x + x^2$$

For $n = 3$, we find:

$$(1+x)^3 = 1 + 3x + \frac{3(2)}{2!}x^2 + \frac{3(2)(1)}{3!}x^3 + \dots = 1 + 3x + 3x^2 + x^3$$

For $n = 1/2$, we find:

$$\begin{aligned}(1+x)^{1/2} &= 1 + \frac{1}{2}x + \frac{\frac{1}{2}(-\frac{1}{2})}{2!}x^2 + \frac{\frac{1}{2}(-\frac{1}{2})(-\frac{3}{2})}{3!}x^3 + \frac{\frac{1}{2}(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})}{4!}x^4 \dots \\ &= 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \frac{5}{128}x^4 + \dots\end{aligned}$$

Finally, we see that:

$$\sqrt{1.0001} = (1+10^{-4})^{1/2} = 1 + \frac{1}{2}10^{-4} - \frac{1}{8}10^{-8} + \frac{1}{16}10^{-12} - \frac{5}{128}10^{-16} + \dots$$

Now, for an alternating series the size of whose terms decreases, we know that if we stop at a certain term, the size of the absolute error is less than the first term we did not include. Thus, we can include the terms through $\frac{1}{16}10^{-12}$ and be sure that the resulting absolute error is less than $\frac{5}{128}10^{-16} < 10^{-15}$. So,

$$\begin{aligned} \sqrt{1.0001} &\approx 1 + \frac{1}{2}10^{-4} - \frac{1}{8}10^{-8} + \frac{1}{16}10^{-12} \\ &= 1 + 0.00005 - 0.00000000125 + 0.0000000000000625 \\ &= 1.0000500000000625 - 0.00000000125 \\ &= 1.0000499987500625 \end{aligned}$$

If you don't know or don't like the alternating series test, you can also use the Taylor error. If we use the approximation

$$\sqrt{1.0001} \approx 1 + \frac{1}{2}10^{-4} - \frac{1}{8}10^{-8} + \frac{1}{16}10^{-12},$$

that means we have used the Taylor series through the $n = 3$ term. Therefore, the error is given by Taylor's Theorem as

$$|Error| = \frac{|f''''(z)|}{4!}(0.0001)^4,$$

for some z between 0 and 0.0001. What is $f''''(z)$? Above, we found that $f'''(x) = n(n-1)(n-2)(1+x)^{n-3}$. So, for $n = 1/2$, $f'''(z) = (1/2)(-1/2)(-3/2)(1+z)^{-5/2} = \frac{3}{8}(1+z)^{-5/2}$. Thus, $f''''(z) = -\frac{5}{16}(1+z)^{-7/2} = -\frac{5}{16(1+z)^{7/2}}$. How big could $|f''''(z)|$ be for z between 0 and 0.0001? Well, since the $(1+z)$ term is in the denominator, the whole expression is maximized when z is as small as it can be, i.e., $z = 0$. So, $|f''''(z)| \leq |\frac{5}{16(1)^{7/2}}| = \frac{5}{16}$. Thus, overall,

$$|Error| \leq \frac{5}{128}(0.0001)^4,$$

which is less than 10^{-15} as we wanted.

2. Example 5 shows that $\ln(1+x) = \sum_{k=1}^{\infty}(-1)^{k-1}\frac{x^k}{k}$ for $-1 < x \leq 1$. How many terms of this series would you need to compute $\ln 2$ accurate to within 10^{-5}? How many terms would be required to get the same accuracy if you first used the analytic trick $\ln 2 = \ln\left(e\frac{2}{e}\right) = 1 + \ln\left(\frac{2}{e}\right)$?

Applying the series, we see that:

$$\ln 2 = \ln(1+1) = \sum_{k=1}^{\infty}(-1)^{k-1}\frac{1}{k}.$$

This is an alternating series the size of whose terms decreases. So, we know that if we stop at the nth term, the absolute error is guaranteed to be less than the $(n+1)$st term. So, to guarantee accuracy to within 10^{-5}, we seek to make the $(n+1)$st term, $\frac{1}{n+1}$, less than 10^{-5}. So, $n+1 > 10^5$, or $n > 10^5 - 1$. (If you don't want to use the alternating series test, you can do a Taylor analysis as at the end of the Problem 1 solution.)

On the other hand, we can exploit the fact that $\ln 2 = \ln(e\frac{2}{e}) = \ln e + \ln\frac{2}{e} = 1 + \ln\frac{2}{e}$. We observe that:

$$\ln\frac{2}{e} = \ln(1 + (\frac{2}{e} - 1)) = \sum_{k=1}^{\infty}(-1)^{k-1}\frac{(\frac{2}{e}-1)^k}{k}.$$

Sadly, this is no longer an alternating series, since $\frac{2}{e} - 1 < 0$, so we instead need to rely on the Taylor error analysis. By this analysis, we know that if we stop at the nth term, then

$$|Error| < \left|\frac{f^{(n+1)}(z)}{(n+1)!}(\frac{2}{e} - 1)^{n+1}\right|$$

for some z between 0 and $\frac{2}{e} - 1$. As shown in Example 5, $f^{(k)}(x) = (-1)^{k-1}(k-1)!(1+x)^{-k}$, so we have that

$$|Error| < \left|\frac{(-1)^n n!(1+z)^{-(n+1)}}{(n+1)!}(\frac{2}{e} - 1)^{n+1}\right| = \frac{1}{(n+1)(1+z)^{n+1}}\left|\frac{2}{e} - 1\right|^{n+1}.$$

Since z is between 0 and $\frac{2}{e} - 1$, we can be sure that $(1+z)^{n+1} \geq (1+\frac{2}{e}-1)^{n+1} = (\frac{2}{e})^{n+1}$, so that

$$|Error| < \frac{1}{(n+1)(\frac{2}{e})^{n+1}}\left|\frac{2}{e} - 1\right|^{n+1} = \frac{1}{n+1}\left|\frac{e}{2}\left(\frac{2}{e} - 1\right)\right|^{n+1} = \frac{1}{n+1}\left|1 - \frac{e}{2}\right|^{n+1}$$

We seek to make this less than 10^{-5}. With some experimentation, we find that $\frac{1}{10}\left|1 - \frac{e}{2}\right|^{10} < 10^{-5}$, so $n = 9$ terms are enough to get the desired accuracy, much less than in the first case.

3. Recall the base-10 `LUCKY-7` system from class: Numbers get stored as $a_1.a_2\cdots a_7 \times 10^c$, where $a_j = 0, 1, 2, \ldots$ or 9, $a_1 \neq 0$ and $-40 < c < 40$. Find numbers a, b, and c so that, in computations on a `LUCKY-7` computer, $a + (b + c) \neq (a + b) + c$.

Let $a = 1 = 1.000000 \times 10^0$ and $b = c = 0.0000004 = 4.000000 \times 10^{-7}$. Each of these is representable in the `LUCKY-7` system as shown by their given scientific notation description.

Now, $a+b = 1.0000004$, but in the `LUCKY-7` system, this would be rounded to 1, since the 4 is the eighth digit after the 1. Similarly, when we add c to this 1, the result will be stored as just 1. So, $(a+b)+c$ is computed to be 1.

On the other hand, $b+c = 0.0000008$, which can be stored exactly in the `LUCKY-7` system (as 8.0000000×10^{-7}). Then, when we add $a = 1$, we get 1.0000008, which gets rounded to 1.000001 when stored in the `LUCKY-7` system. So, $a+(b+c)$ is computed to be 1.000001, which is not equal to $(a+b)+c$.

4. Later in the course we will see that $f''(a)$ is often approximated as follows:

$$f''(a) \approx \frac{f(a+h) - 2f(a) + f(a-h)}{h^2},$$

for some small h. Using the Taylor series expansions of $f(a+h)$ and $f(a-h)$ about a, verify that the leading-order term of this expression is, in fact, $f''(a)$, and determine the order of the approximation (i.e., is it correct to $O(h)$, $O(h^2)$, etc.)?

By the Taylor series expansion,

$$f(a+h) = f(a) + f'(a)h + \frac{f''(a)}{2}h^2 + \frac{f'''(a)}{6}h^3 + \frac{f''''(a)}{24}h^4 + \cdots .$$

Inserting $-h$ for h, we find:

$$\begin{aligned} f(a-h) &= f(a) + f'(a)(-h) + \frac{f''(a)}{2}(-h)^2 + \frac{f'''(a)}{6}(-h)^3 + \frac{f''''(a)}{24}(-h)^4 \cdots \\ &= f(a) - f'(a)h + \frac{f''(a)}{2}h^2 - \frac{f'''(a)}{6}h^3 + \frac{f''''(a)}{24}h^4 + \cdots . \end{aligned}$$

Adding these results together, we find:

$$f(a+h) + f(a-h) = 2f(a) + f''(a)h^2 + \frac{f''''(a)}{12}h^4 + \cdots .$$

Plugging this into the given quotient, we find:

$$\begin{aligned} \frac{f(a+h) - 2f(a) + f(a-h)}{h^2} &= \frac{\left(2f(a) + f''(a)h^2 + \frac{f''''(a)}{12}h^4 + \cdots\right) - 2f(a)}{h^2} \\ &= \frac{f''(a)h^2 + \frac{f''''(a)}{12}h^4 + \cdots}{h^2} \\ &= f''(a) + \frac{f''''(a)}{12}h^2 + \cdots \end{aligned}$$

So, the leading-order term is indeed $f''(a)$, which is why this quotient is a good approximation for $f''(a)$. The approximation is $O(h^2)$, as evidenced by the fact that the first-order correction term $\frac{f''''(a)}{12}h^2$ is a constant times h^2.

5. Consider implementing the approximation from Problem 4 on a LUCKY-7 computer for the function $f(x) = \sqrt{x}$ at $a = 1$. Imagine that the computer can determine $f(x)$ exactly for any x, but then must store it as the nearest LUCKY-7 number before plugging it into the quotient used to approximate $f''(1)$.

Compute the exact value of $f''(1)$. Next, consider a range of h values between 10^{-3} and 1 and compute the quotient *in the LUCKY-7 number system*. **How does the accuracy of the approximation depend on h in this range? Why does it depend on h in this way, in terms of the types of errors we discussed in class? Estimate the value of h in this range that gives the best approximation to $f''(1)$.**

Since $f(x) = \sqrt{x} = x^{1/2}$, we have $f'(x) = \frac{1}{2}x^{-1/2}$ and $f''(x) = -\frac{1}{4}x^{-3/2}$, and therefore $f''(1) = -\frac{1}{4}$. For the rest, I made a Mathematica notebook to simulate the computations. You can download this notebook on the solutions Web page.

First, here's a function that computes the LuckySeven version of x. It ignores underflow/overflow (i.e., it returns $a_1.a_2a_3a_4a_5a_6a_7 \times 10^c$, regardless of whether c is less than -40 or greater than 40.

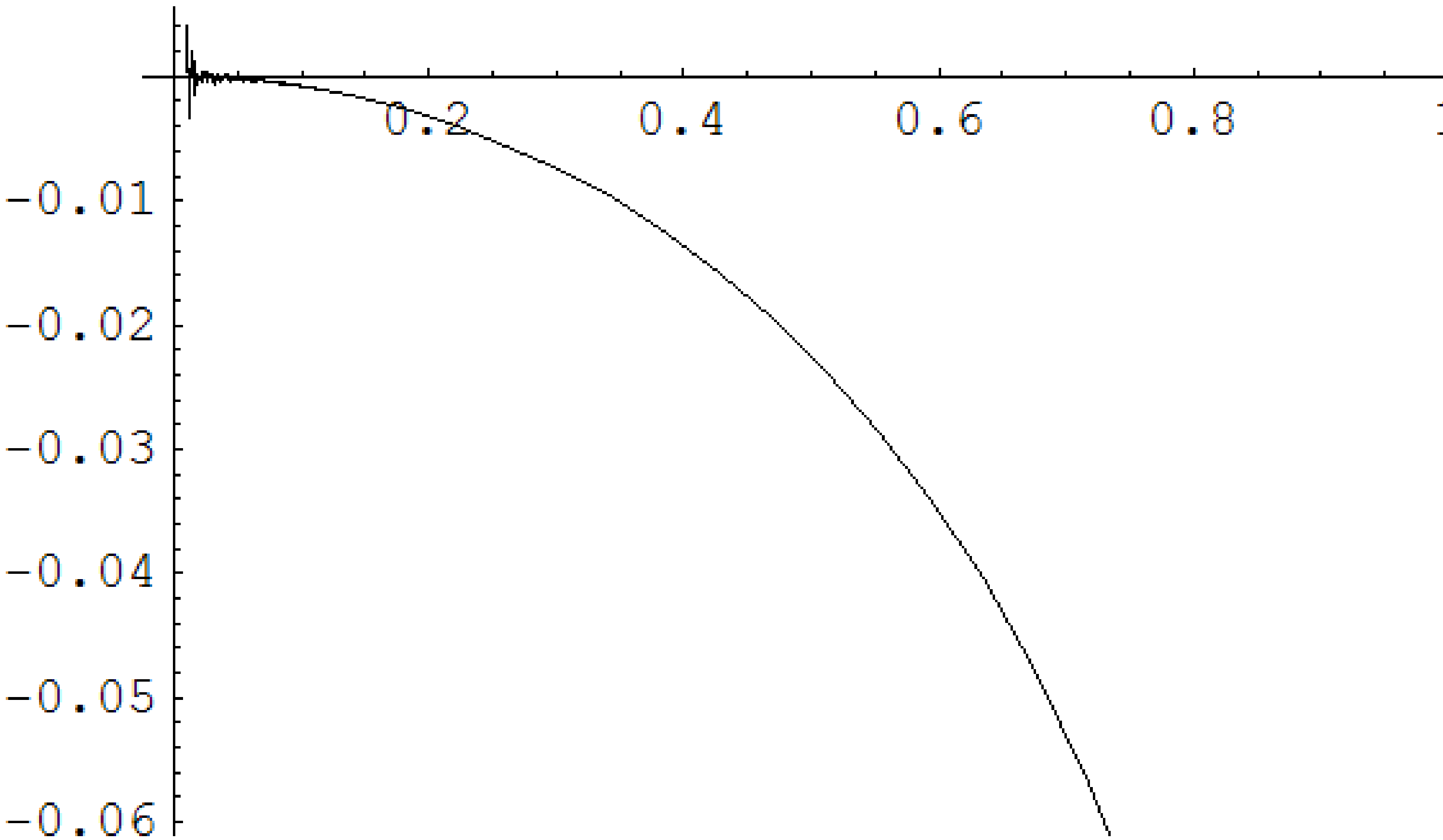
```
LuckySeven[x_] :=
  Block[{},
   If[x ≠ 0,
    c = 0;
    y = Abs[x];
    While[y < 1, c = c - 1; y = y * 10];
    While[y > 10, c = c + 1; y = y / 10];
    z = Floor[y * 1000000`25 + 1 / 2] / 100000
    z = 0];
   Sign[x] * z * 10 ^ c
  ];
```

Now here's a function that computes the error in the LuckySeven version of the approximation to the second derivative of $\sqrt{x}$ at $x = 1$.

```
f[h_] :=
  Block[{},
   a = LuckySeven[Sqrt[1.0`25 + h]];
   b = LuckySeven[Sqrt[1.0`25 - h]];
   c = LuckySeven[a - 2.0`25];
   d = LuckySeven[c + b];
   e = LuckySeven[h^2];
   LuckySeven[d / e] - (-0.25`25)
  ];
```

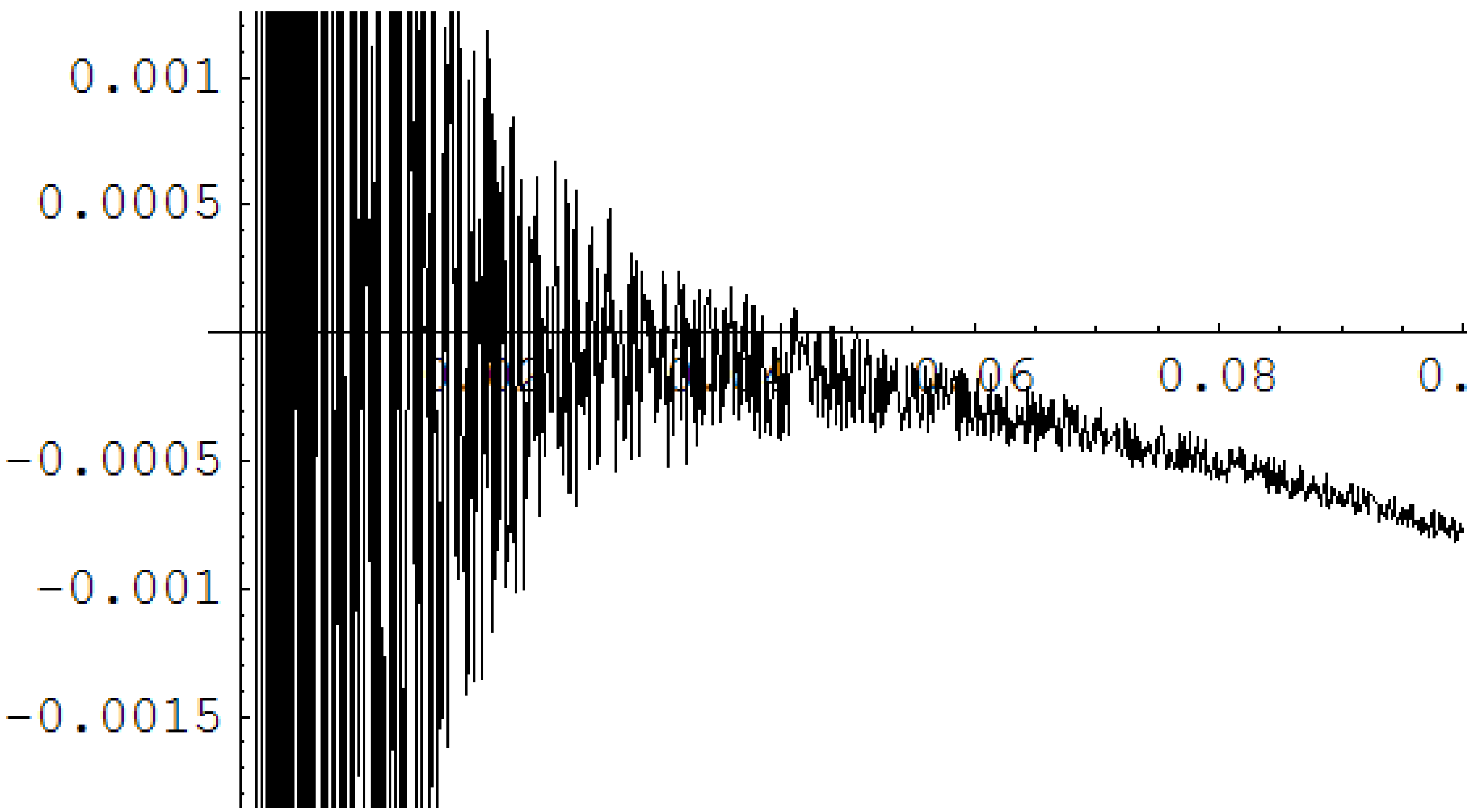
Here's a plot of the error versus h, for h between 0.01 and 1.

```
Plot[f[h], {h, 0.01, 1.0}]
```



Looks like a nice $O(h^2)$ improvement as predicted by the Taylor expansion. What's up with the wiggles for small h, though? Let's make a plot for h between $h = 0.001$ and $h = 0.1$.

```
Plot[f[h], {h, 0.001, 0.1}]
```



Whoa – something different is going on now. Roundoff error is kicking in around $h = 0.05$, which makes sense, since the formula for approximating the second derivative involves subtracting some nearly equal quantities and then dividing by a small number. It seems like we should choose $h = 0.05$, since that's where the spread of errors is smallest and centered around zero. For $h > 0.05$, the error is consistently negative and getting larger as h increases. For $h < 0.05$, the rapid oscillations in the error get larger and larger as h decreases.