

Math 116 Fall 2003—Solutions to Practice Problems for Final Exam

1. A hat contains paper slips numbered 1 through 20. Three are drawn at random. What is the probability that they are consecutive (e.g., 5, 6, 7)?

The sample space S is all sets of 3 slips from the 20, so that $N(S) = \binom{20}{3} = 1140$. The event A in question is all sets of 3 consecutive slips from the 20, so that $N(A) = 18$ ($\{1, 2, 3\}$ through $\{18, 19, 20\}$). Thus, $P(A) = \frac{18}{1140} = 0.016$.

2. Suppose that ten percent of cats have a hyperthyroid condition. The vet offers a test for this condition. 95% of cats with the condition will test positive, while 10% of cats without the condition will test positive. If a cat is chosen at random and tested, what is the probability that the test will be positive? Given that a cat tests positive, what is the probability that the cat has the condition?

We are told that $P(\text{hypthy}) = 0.1$, $P(\text{positive}|\text{hypthy}) = 0.95$, and $P(\text{positive}|\text{hypthy}') = 0.1$.

So,

$$\begin{aligned} P(\text{positive}) &= P(\text{positive} \cap \text{hypthy}) + P(\text{positive} \cap \text{hypthy}') \\ &= P(\text{positive}|\text{hypthy})P(\text{hypthy}) + P(\text{positive}|\text{hypthy}')P(\text{hypthy}') \\ &= (0.95)(0.1) + (0.1)(0.9) = 0.185 \end{aligned}$$

And,

$$\begin{aligned} P(\text{hypthy}|\text{positive}) &= \frac{P(\text{hypthy} \cap \text{positive})}{P(\text{positive})} \\ &= \frac{P(\text{positive}|\text{hypthy})P(\text{hypthy})}{P(\text{positive})} = \frac{(0.95)(0.1)}{0.185} = 0.514 \end{aligned}$$

3. A box contains 5 red balls and 3 blue balls. I draw 2 balls from the box and put one on my head and one on my nose. Given that the ball on my head is blue, what is the probability that I drew 2 blue balls? Given that at least one of the balls I drew was blue, what is the probability that I drew 2 blue balls?

Now, by definition,

$$P(\text{both blue}|\text{head blue}) = \frac{P(\text{both blue} \cap \text{head blue})}{P(\text{head blue})}.$$

Note, however, that the event $\text{both blue} \cap \text{head blue}$ can be stated more simply as just the event that both balls are blue. So,

$$P(\text{both blue}|\text{head blue}) = \frac{P(\text{both blue})}{P(\text{head blue})}.$$

Now, $P(\text{head blue}) = \frac{3}{8}$, since we must select one of three blue balls from a pool of eight, and $P(\text{both blue}) = \frac{3}{8} \cdot \frac{2}{7} = \frac{3}{28}$, since we must select one of three blue balls from a pool of eight, and then, following that, select one of the remaining two blue balls from a pool of seven. So,

$$P(\text{both blue}|\text{head blue}) = \frac{3/28}{3/8} = \frac{8}{28} = \frac{2}{7}.$$

(Or, we could have gotten this more simply by simply noting that, once we know the head ball is blue, the chance that the other one is blue, and hence both are blue, is just $2/7$.)

Now, by definition,

$$P(\text{both blue}|\text{at least one blue}) = \frac{P(\text{both blue} \cap \text{at least one blue})}{P(\text{at least one blue})}.$$

Note, however, that the event $\text{both blue} \cap \text{at least one blue}$ can be stated more simply as just the event that both balls are blue. So,

$$P(\text{both blue}|\text{at least one blue}) = \frac{P(\text{both blue})}{P(\text{at least one blue})}.$$

Now, $P(\text{at least one blue}) = 1 - P(\text{both red}) = 1 - \frac{5}{8} \cdot \frac{4}{7} = 1 - \frac{5}{14} = \frac{9}{14}$, by the same logic as above for computing $P(\text{both blue})$. Further, from above, we have $P(\text{both blue}) = \frac{3}{28}$, So,

$$P(\text{both blue}|\text{at least one blue}) = \frac{3/28}{9/14} = \frac{1}{6}.$$

4. p. 140, # 91: After shuffling a deck of 52 cards, a dealer deals out five. Let X be the number of suits represented in the hand. Compute the pmf of X , and compute μ and σ .

The sample space S is all sets of five cards chosen from 52, so $N(S) = \binom{52}{5} = 2598960$.

Consider the event $A = \{X = 1\}$. There are four choices for the suit, and then, given that choice, there are $\binom{13}{5} = 1287$ choices for the ranks of the five cards, so $N(A) = (4)(1287) = 5148$, and thus $P(X = 1) = \frac{5148}{2598960} = 0.002$.

Now consider the event $B = \{X = 2\}$. There are four choices for the suit with more cards, and then three choices for the suit with fewer cards. Then, once we know these two suits, we could have 4 of the first suit and one of the other, or 3 of the first suit and 2 of the other. In the first case, we have $\binom{13}{4} = 715$ choices for the 4 cards, and 13 choices for the other, and thus $(715)(13) = 9295$ choices overall. In the second case, we have $\binom{13}{3} = 286$ choices for the 3 cards, and $\binom{13}{2} = 78$ choices for the other two, and thus $(286)(78) = 22308$ choices overall. So, $N(B) = (4)(3)(9295 + 22308) = 379236$, and thus $P(X = 2) = \frac{379236}{2598960} = 0.146$.

Now consider the event $C = \{X = 4\}$ (I'll get $X = 3$ by subtraction at the end, since it seems hardest). There are four choices for the suit with two cards. Then, there are $\binom{13}{2} = 78$ choices for the pair of cards in this suit, and 13 choices in each of the other suits, for a total of $N(C) = 4(78)(13)(13)(13) = 685464$, so that $P(X = 4) = \frac{685464}{2598960} = 0.264$.

Finally $P(X = 3) = 1 - P(X = 1) - P(X = 2) - P(X = 4) = 0.588$.

Now, $\mu = E(X) = (1)(0.002) + (2)(0.146) + (3)(0.588) + (4)(0.264) = 3.114$.

Then, $E(X^2) = (1^2)(0.002) + (2^2)(0.146) + (3^2)(0.588) + (4^2)(0.264) = 10.102$.

Thus, $\sigma^2 = E(X^2) - [E(X)]^2 = 0.405$, and so $\sigma = \sqrt{0.405} = 0.636$.

5. A business offers the following health care plan. You deposit \$ $k/2$ at the beginning of the year, and the business also deposits \$ $k/2$. You can use this \$ k all year for health care costs. If your yearly health care costs are less than \$ k , you don't get any of your deposit

back. If your yearly costs are more than \$ k , you pay all costs above \$ k out of your pocket.

Let X be the random variable representing your health care costs for the year, and suppose it has the following distribution:

x	\$ 0	\$ 1000	\$ 2000	\$ 4000	\$ 7000	\$ 10000
$P(X = x)$	0.2	0.25	0.25	0.15	0.1	0.05

Would it be better to deposit $k = 1000$ or $k = 2000$ dollars at the beginning of the year (i.e., which choice gives the lower expected total cost).

Suppose $k = 1000$. Let's make a table of the possible values of our total cost and their respective probabilities. If our health care costs are less than or equal to 2000 dollars, then we'll spend 1000 dollars; otherwise, if our costs are D dollars with $D > 2000$, then we'll spend $1000 + (D - 2000) = D - 1000$:

$Cost$	\$ 1000	\$ 3000	\$ 6000	\$ 9000
$P(Cost)$	$0.2 + 0.25 + 0.25$	0.15	0.1	0.05

So, in this case

$$E(Cost) = (1000)(0.7) + (3000)(0.15) + (6000)(0.1) + (9000)(0.05) = 2200.$$

Now suppose $k = 2000$. Let's make a table of the possible values of our total cost and their respective probabilities. If our health care costs are less than or equal to 4000 dollars, then we'll spend 2000 dollars; otherwise, if our costs are D dollars with $D > 4000$, then we'll spend $2000 + (D - 4000) = D - 2000$:

$Cost$	\$ 2000	\$ 5000	\$ 8000
$P(Cost)$	$0.2 + 0.25 + 0.25 + 0.15$	0.1	0.05

So, in this case

$$E(Cost) = (2000)(0.85) + (5000)(0.1) + (8000)(0.05) = 2600.$$

6. Suppose that 10% of Americans have blue eyes. If you choose 15 Americans at random, what is the probability that between three and five have blue eyes? What is the probability that four or more have blue eyes?

Let X be the number of Americans from the 15 chosen with blue eyes. Then X is binomial with $n = 15$ and $p = 0.1$. The first probability is

$$\begin{aligned} P(3 \leq X \leq 5) &= \binom{15}{3} (0.1)^3 (0.9)^{12} + \binom{15}{4} (0.1)^4 (0.9)^{11} + \binom{15}{5} (0.1)^5 (0.9)^{10} \\ &= 0.1285 + 0.0428 + 0.0105 = 0.1818. \end{aligned}$$

The second probability is:

$$\begin{aligned} P(X \geq 4) &= 1 - P(X \leq 3) \\ &= 1 - \binom{15}{3} (0.1)^3 (0.9)^{12} - \binom{15}{2} (0.1)^2 (0.9)^{13} \\ &\quad - \binom{15}{1} (0.1)^1 (0.9)^{14} - \binom{15}{0} (0.1)^0 (0.9)^{15} \\ &= 1 - 0.1285 - 0.2669 - 0.3432 - 0.2059 = 0.0555 \end{aligned}$$

7. p. 199, # 93: The random variable Y (the distance from the left end of a 12-in bar where a snap occurs) has density

$$f(y) = \begin{cases} \frac{y}{24} \left(1 - \frac{y}{12}\right) & 0 \leq y \leq 12 \\ 0 & \text{otherwise} \end{cases}.$$

Compute the cdf of Y , $P(Y \leq 4)$, $P(Y > 6)$, $P(4 \leq Y \leq 6)$, $E(Y)$, $E(Y^2)$, $V(Y)$, the probability that the break point occurs more than 2 in from the expected break point, and the expected length of the shorter segment when the break occurs.

We'll denote the cdf of Y by $F(x)$. For $x < 0$, $F(x) = 0$, and for $x > 12$, $F(x) = 1$. For $0 \leq x \leq 12$,

$$\begin{aligned} F(x) &= \int_0^x f(y) dy \\ &= \int_0^x \left(\frac{y}{24} - \frac{y^2}{288} \right) dy \\ &= \left(\frac{y^2}{48} - \frac{y^3}{864} \right) \Big|_0^x = \frac{x^2}{48} - \frac{x^3}{864} \end{aligned}$$

Now $P(Y \leq 4) = F(4) = 0.259$, or you could use an integral,

$$\begin{aligned} P(Y \leq 4) &= \int_0^4 \left(\frac{y}{24} - \frac{y^2}{288} \right) dy \\ &= \left(\frac{y^2}{48} - \frac{y^3}{864} \right) \Big|_0^4 = \frac{4^2}{48} - \frac{4^3}{864} = 0.259. \end{aligned}$$

Similarly, $P(Y > 6) = 1 - P(Y \leq 6) = 1 - F(6) = 0.5$, or you could use an integral:

$$\begin{aligned} P(Y > 6) &= \int_6^{12} \left(\frac{y}{24} - \frac{y^2}{288} \right) dy \\ &= \left(\frac{y^2}{48} - \frac{y^3}{864} \right) \Big|_6^{12} = \left(\frac{12^2}{48} - \frac{12^3}{864} \right) - \left(\frac{6^2}{48} - \frac{6^3}{864} \right) = 0.5. \end{aligned}$$

Now, $P(4 \leq Y \leq 6) = 1 - P(Y < 4) - P(Y > 6) = 1 - 0.259 - 0.5 = 0.241$.

Next,

$$\begin{aligned} E(Y) &= \int_0^{12} y \left(\frac{y}{24} - \frac{y^2}{288} \right) dy \\ &= \int_0^{12} \left(\frac{y^2}{24} - \frac{y^3}{288} \right) dy \\ &= \left(\frac{y^3}{72} - \frac{y^4}{1152} \right) \Big|_0^{12} = \frac{12^3}{72} - \frac{12^4}{1152} = 6 \end{aligned}$$

Similarly,

$$\begin{aligned} E(Y^2) &= \int_0^{12} y^2 \left(\frac{y}{24} - \frac{y^2}{288} \right) dy \\ &= \int_0^{12} \left(\frac{y^3}{24} - \frac{y^4}{288} \right) dy \\ &= \left(\frac{y^4}{96} - \frac{y^5}{1440} \right) \Big|_0^{12} = \frac{12^4}{96} - \frac{12^5}{1440} = 43.2, \end{aligned}$$

and so, $V(Y) = E(Y^2) - [E(Y)]^2 = 43.2 - 6^2 = 7.2$.

The probability that the break point occurs more than 2 from the expected point is:

$$1 - P(4 \leq Y \leq 8) = 1 - (F(8) - F(4)) = 1 - 0.481 = 0.519.$$

Now, what is this function “length of shorter segment”? Well, if $Y \leq 6$, then it’s just Y , whereas if $Y > 6$, then it’s $12 - Y$. So, we need to find the expected value of this function $h(Y)$, which is:

$$\begin{aligned}
 E(h(Y)) &= \int_0^{12} h(y)f(y)dy \\
 &= \int_0^6 y \left(\frac{y}{24} - \frac{y^2}{288} \right) dy + \int_6^{12} (12 - y) \left(\frac{y}{24} - \frac{y^2}{288} \right) dy \\
 &= \int_0^6 \left(\frac{y^2}{24} - \frac{y^3}{288} \right) dy + \int_6^{12} \left(\frac{y}{2} - \frac{y^2}{24} - \frac{y^2}{24} + \frac{y^3}{288} \right) dy \\
 &= \int_0^6 \left(\frac{y^2}{24} - \frac{y^3}{288} \right) dy + \int_6^{12} \left(\frac{y}{2} - \frac{y^2}{12} + \frac{y^3}{288} \right) dy \\
 &= \left(\frac{y^3}{72} - \frac{y^4}{1152} \right) \Big|_0^6 + \left(\frac{y^2}{4} - \frac{y^3}{36} + \frac{y^4}{1152} \right) \Big|_6^{12} \\
 &= \left(\frac{6^3}{72} - \frac{6^4}{1152} \right) + \left(\frac{12^2}{4} - \frac{12^3}{36} + \frac{12^4}{1152} \right) - \left(\frac{6^2}{4} - \frac{6^3}{36} + \frac{6^4}{1152} \right) \\
 &= 1.875 + 6 - 4.125 = 3.75
 \end{aligned}$$

8. p. 199, # 96: The breakdown voltage of a randomly chosen diode of a certain type is normal with $\mu = 40V$ and $\sigma = 1.5V$. What is the probability that the voltage is between 39 and 42 V? What value is such that only 15 % of diodes exceed that value? If four diodes are selected independently, what is the probability that at least one has a voltage exceeding 42 V?

We are asked to find $P(39 \leq X \leq 42)$, which equals $P(\frac{39-40}{1.5} \leq \frac{X-40}{1.5} \leq \frac{42-40}{1.5})$, or $P(-0.667 \leq \frac{X-40}{1.5} \leq 1.333)$. Now, $\frac{X-40}{1.5}$ is a standard normal, call it Z , so we are seeking $P(-0.667 \leq Z \leq 1.333) = \Phi(1.333) - \Phi(-0.667)$, which we can see from the table equals $0.9082 - 0.2514 = 0.6568$.

We are asked to find c so that $P(X \geq c) = 0.15$. In other words, we want $P(\frac{X-40}{1.5} \geq \frac{c-40}{1.5}) = 0.15$. Now, $\frac{X-40}{1.5}$ is a standard normal, call it Z , so we are seeking $P(c \leq Z) = 0.15$. By symmetry of the standard normal, we can see that this is the same as $P(Z \leq -c) = 0.15$, i.e., $\Phi(-c) = 0.15$. From the table of Φ , we can see that $-c = -1.04$, i.e., $c = 1.04$.

For the final question, we first determine the probability that one diode’s voltage exceeds 42. From the first part, we know that this is equivalent to $P(Z \geq 1.333)$ where Z is a standard normal, which equals $P(Z \leq -1.333) = 0.092$. Now, the

probability that at least one of the four chosen diodes exceeds 42 V is one minus the probability that all of them do not exceed 42 V, i.e. $1 - (1 - 0.092)^4 = 0.3202$.

9. p. 199, # 100 bcde, and compute the median: Given a random variable X (reaction time in seconds) with density

$$f(x) = \begin{cases} \frac{3}{2x^2} & 1 \leq x \leq 3 \\ 0 & \text{otherwise} \end{cases}$$

what is the probability that reaction time is at most 2.5 sec? Between 1.5 and 2.5 sec? Compute the expected reaction time and its standard deviation and median. If an individual takes more than 1.5 sec to react, a light comes on and stays on either until one further second has elapsed or until the person reacts (whichever happens first). Determine the expected amount of time that the light remains lit.

The probability that reaction time is at most 2.5 seconds is

$$P(X \leq 2.5) = \int_1^{2.5} 1.5x^{-2} dx = -1.5x^{-1} \Big|_1^{2.5} = -\frac{1.5}{2.5} + 1.5 = -0.6 + 1.5 = 0.9.$$

The probability that reaction time is between 1.5 and 2.5 seconds is

$$P(1.5 \leq X \leq 2.5) = \int_{1.5}^{2.5} 1.5x^{-2} dx = -1.5x^{-1} \Big|_{1.5}^{2.5} = -\frac{1.5}{2.5} + \frac{1.5}{1.5} = -0.6 + 1.0 = 0.4.$$

The expected reaction time is:

$$E(X) = \int_1^3 x(1.5x^{-2}) dx = \int_1^3 (1.5x^{-1}) dx = 1.5 \ln x \Big|_1^3 = 1.5 \ln 3 - 0 = 1.648.$$

In addition:

$$E(X^2) = \int_1^3 x^2(1.5x^{-2}) dx = \int_1^3 1.5 dx = 1.5x \Big|_1^3 = 1.5(3) - 1.5 = 3,$$

so that

$$V(X) = E(X^2) - [E(X)]^2 = 3 - 1.648^2 = 0.284,$$

and

$$\sigma_X = \sqrt{0.284} = 0.533.$$

To compute the median M , we solve the equation:

$$\int_1^M (1.5x^{-2}) dx = 0.5,$$

which means that

$$-1.5x^{-1}|_1^M = 0.5,$$

or

$$-1.5M^{-1} + 1.5 = 0.5,$$

or

$$-1.5M^{-1} = -1 \implies -1.5 = -M \implies M = 1.5$$

Now, if the reaction time is less than 1.5 sec, the light is never on, so the time that it is lit is zero. If the reaction time x is between 1.5 sec and 2.5 sec, the light is on for $x - 1.5$ sec. If the reaction time x is greater than 2.5 sec, the light is on for 1 sec. So, if $h(X)$ is the time the light is lit:

$$h(x) = \begin{cases} 0 & x \leq 1.5 \\ x - 1.5 & 1.5 < x \leq 2.5 \\ 1 & 2.5 < x \leq 3 \end{cases}$$

So,

$$\begin{aligned} E(h(X)) &= \int_1^3 h(x)f(x)dx \\ &= \int_1^{1.5} 0dx + \int_{1.5}^{2.5} (x - 1.5)(1.5x^{-2})dx + \int_{2.5}^3 (1)(1.5x^{-2})dx \\ &= \int_{1.5}^{2.5} (1.5x^{-1} - 2.25x^{-2})dx + \int_{2.5}^3 (1.5x^{-2})dx \\ &= (1.5 \ln x + 2.25x^{-1})_{1.5}^{2.5} + (-1.5x^{-1})_{2.5}^3 \\ &= (1.5 \ln 2.5 + \frac{2.25}{2.5}) - (1.5 \ln 1.5 + \frac{2.25}{1.5}) + (-\frac{1.5}{3} + \frac{1.5}{2.5}) \\ &= 0.266 \end{aligned}$$