

## HSSI 2007 Math —HW #1 Solutions

Problems from §1.1:

**10. Match the graphs in the given figure with the following equations (Note that the  $x$  and  $y$  scales may be unequal).**

(a)  $y = x - 5$  is (V): a line with slope one (45 degree tilt) and  $y$ -intercept  $-5$  (below the  $x$ -axis).

(b)  $-3x + 4 = y$  is (IV): a line with negative slope and positive  $y$ -intercept

(c)  $5 = y$  is (I): a horizontal line

(d)  $y = -4x - 5$  is (VI): a line with negative slope and negative  $y$ -intercept

(e)  $y = x + 6$  is (II): a line with positive slope and positive  $y$ -intercept

(f)  $y = x/2$  is (III): a line with positive slope and zero  $y$ -intercept

**12. Find the equation of the line through the point  $(2, 1)$  which is perpendicular to the line  $y = 5x - 3$ .**

The slope of the line  $y = 5x - 3$  is 5. The slope of a line perpendicular to this line must be  $-1/5$  (the negative reciprocal). Therefore, the equation of the new line would be  $y = (-1/5)x + b$ . Now we need to find  $b$ . The line must pass through  $(2, 1)$ , so that when  $x = 2$ , we have  $y = 1$ , which means  $1 = (-1/5)(2) + b$ , or  $1 = (-2/5) + b$ , or  $b = 7/5$ . So, the equation of the line is  $y = (-1/5)x + (7/5)$ .

**30. Residents of the town of Maple Grove who are connected to the municipal water supply are billed a fixed amount yearly plus a charge for each cubic foot of water used. A household using 1000 cubic feet was billed \$ 90, while one using 1600 cubic feet was billed \$ 105.**

(a) **What is the charge per cubic foot?**

The function that relates dollars  $y$  to cubic feet used  $x$  is a linear function  $y = mx + b$ , given how the problem describes the cost system. The charge per cubic foot is just the slope  $m$  of the linear function. We know that this can be computed as  $\Delta y / \Delta x$ . We are given two  $(x, y)$  values, namely  $(1000, 90)$  and  $(1600, 105)$ . Therefore,  $\Delta y = 105 - 90 = 15$  and  $\Delta x = 1600 - 1000 = 600$ , so

that  $m = 15/600 = 0.025$ . Thus, it cost \$ 0.025, or 2.5 cents, per cubic foot.

**(b) Write an equation for the total cost as a function of cubic feet used.**

We just have to find  $b$ . We know already that  $y = 0.025x + b$ . Plugging in the first of our given points, we have  $90 = 0.025(1000) + b$ , or  $90 = 25 + b$ , or  $65 = b$ . So, Cost =  $0.025$  (cubic feet used) + 65.

**(c) How many cubic feet used would lead to a cost of \$ 130?**

We seek  $x$  so that  $130 = 0.025x + 65$ , or, subtracting 65 from each side,  $65 = 0.025x$ , or, dividing both sides by 0.025, we find  $2600 = x$ . So, you'd have to use 2600 cubic feet.

**31. The graph of Fahrenheit temperature  $^{\circ}F$  as a function of Celsius temperature  $^{\circ}C$  is a line. You know that  $212^{\circ}F$  and  $100^{\circ}C$  both represent the temperature at which water boils. Similarly,  $32^{\circ}F$  and  $0^{\circ}C$  both represent the temperature at which water freezes.**

**(a) What is the slope of the graph?**

The function that relates  $F$  to  $C$  is a linear function  $F = mC + b$ . The slope  $m$  can be computed as  $\Delta F/\Delta C$ . We are given two  $(F, C)$  values, namely  $(212, 100)$  and  $(32, 0)$ . Therefore,  $\Delta F = 212 - 32 = 180$  and  $\Delta C = 100 - 0 = 100$ , so that  $m = 180/100 = 1.8$ .

**(b) What is the equation of the line?**

We just have to find  $b$ . We know already that  $F = 1.8C + b$ . Plugging in the second of our given points, we have  $32 = 1.8(0) + b$ , or  $32 = b$ . So,  $F = 1.8C + 32$ .

**(c) What Fahrenheit temperature corresponds to  $20^{\circ}C$ ?**

It's  $F = 1.8(20) + 32 = 68$  degrees.

**(d) What temperature is the same number of degrees in both scales?**

We need  $F = C$ , so we replace  $F$  by  $C$  and solve for  $C$ : we have  $C = 1.8C + 32$ , or, subtracting  $1.8C$  from both sides,  $-0.8C = 32$ . Dividing both sides by  $-0.8$ , we find  $C = -40$ . So,  $-40^{\circ}F$  and  $-40^{\circ}C$  are the same.

**34. The table in the book gives average weight  $w$ , in pounds, of American men in their sixties for various heights  $h$  in inches.**

**(a) How do you now that the data in this table could represent a linear function?**

Every increase in  $h$  by one inch gives a corresponding increase of  $w$  by 5 pounds.

**(b) Find weight  $w$  as a linear function of height  $h$ . What is the slope of the line? What are its units?**

The function that relates  $w$  to  $h$  is a linear function  $w = mh + b$ . The slope  $m$  can be computed as  $\Delta w / \Delta h$ . We are given eight  $(w, h)$  values in the table. Choosing the first two,  $\Delta w = 171 - 166 = 5$  and  $\Delta h = 69 - 68 = 1$ , so that  $m = 5/1 = 5$ . Since the units of  $w$  are pounds and the units of  $h$  are inches, the units of the slope are pounds per inch.

Now, we just have to find  $b$ . We know already that  $w = 5h + b$ . Plugging in the first of our given points, we have  $166 = 5(68) + b$ , or  $166 = 340 + b$ , or  $-174 = b$ . So, the equation is  $w = 5h - 174$ .

**(c) Find height  $h$  as a linear function of weight  $w$ . What is the slope of the line? What are its units?**

The function that relates  $h$  to  $w$  is a linear function  $h = mw + b$ . The slope  $m$  can be computed as  $\Delta h / \Delta w$ . We are given eight  $(h, w)$  values in the table. Choosing the first two,  $\Delta w = 171 - 166 = 5$  and  $\Delta h = 69 - 68 = 1$ , so that  $m = 1/5$ . Since the units of  $w$  are pounds and the units of  $h$  are inches, the units of the slope are inches per pound.

Now, we just have to find  $b$ . We know already that  $h = (1/5)w + b$ . Plugging in the first of our given points, we have  $68 = (1/5)(166) + b$ , or  $68 = 33.2 + b$ , or  $b = 34.8$ . So, the equation is  $h = (1/5)w + 34.8$ .

**35. The demand function for a certain product  $q = D(p)$  is linear, where  $p$  is the price per item in dollars and  $q$  is the quantity demanded. If  $p$  increases by \$ 5, then  $q$  drops by two units. In addition, 100 items are purchased if the price is \$ 550.**

**(a) Find a formula for  $q$  as a linear function of  $p$ , and of  $p$  as a linear function of  $q$ .**

The function that relates  $p$  to  $q$  is a linear function  $p = mq + b$ . The slope  $m$  can be computed as  $\Delta p / \Delta q$ . We are given  $\Delta p$  and a corresponding  $\Delta q$  in the problem:  $\Delta p = 5$  and  $\Delta q = -2$ , so that  $m = -5/2$ .

Now, we just have to find  $b$ . We know already that  $p = (-5/2)q + b$ . Plugging in our given point  $(p, q) = (550, 100)$ , we have  $550 = (-5/2)(100) + b$ , or  $550 = -250 + b$ , or  $b = 800$ . So, the equation is  $p = (-5/2)q + 800$ .

To find the equation for  $p$  in terms of  $q$ , we just solve this equation for  $q$ . First we subtract 800 from each side to find  $p - 800 = (-5/2)q$ . Then we multiply each side by  $-2/5$  to find  $(-2/5)p - (-2/5)(800) = q$ , or  $(-2/5)p + 320 = q$ . So, the equation is  $q = (-2/5)p + 320$ .

**(b) Draw a graph with  $q$  on the horizontal axis.**

You should have a line with negative slope (decreasing 5 units for every 2 units in the  $x$  direction), with  $y$ -intercept at  $(0, 800)$ .

Problems from §1.2:

**6. An air freshener starts with 30 grams and evaporates. In each of the following cases, write a formula for the quantity  $Q$  (in grams) of air-freshener remaining  $t$  days after the start and sketch a graph of the function.**

**(a) for the case of a decrease of 2 grams per day**

This describes a linear function, with slope  $-2$ . We know that when  $t = 0$ , we have  $Q = 30$ , so we have a  $y$ -intercept of 30. Thus, the formula is  $Q = -2t + 30$ . The graph is a line with negative slope and  $y$ -intercept 30 (and passing through  $(15, 0)$  as well).

**(b) for the case of a decrease of 12% per day**

This describes an exponential function. After one day,  $Q$  will be 0.88 times what we started with, and each additional day, we multiply by another factor of 0.88. So, the formula is  $Q = 30(0.88)^t$ . The graph has an exponential decay shape, starting at  $(0, 30)$  and decaying to an asymptote of the positive  $x$ -axis.

**11. Identify the  $x$ -intervals in the picture on which the function shown is**

**(a) Increasing and concave up**

(D,E) and (H,I)

**(b) Increasing and concave down**

(A,B) and (E,F)

**(c) Decreasing and concave up**

(C,D) and (G,H)

**(d) Decreasing and concave down**

(B,C) and (F,G)

**12. In 1999, the world's population reached 6 billion and was increasing at a rate of 1.3% per year. Assume that this growth rate remains constant.**

**(a) Write a formula for the world population (in billions) as a function of the number of years since 1999.**

This is standard exponential growth  $P = P_0 a^t$ . We are told the initial population  $P_0 = 6$ . In addition, the growth rate tells us that  $a = 1.013$ . So,  $P = 6(1.013)^t$ .

**(b) Estimate the population in 2020.**

This is 21 years after 1999, so  $P = 6(1.013)^{21} = 7.87$  billion.

**(c) Sketch a graph of the population.**

You should have a standard exponential growth curve starting at (1999, 6). We know now that it passes through (2020, 7.87). By eye, you might estimate the doubling time as somewhere around 50 years. If you want to be more accurate, you could solve (using logarithms):  $12 = 6(1.013)^t$ , which implies, dividing both sides by 6, that  $2 = (1.013)^t$ . Taking  $\ln$  of each side, and using a rule of logarithms, we find  $\ln 2 = t \ln(1.013)$ , or  $t = (\ln 2)/(\ln 1.013) = 53.7$  years.

**18. Give a possible formula for the curve shown.**

We want exponential decay asymptoting to the positive  $x$ -axis, so we have

$y = Ca^x$ . As  $x$  changes from  $-1$  to  $1$  (i.e.,  $x$  grows by  $2$ ), we notice that  $y$  decreases from  $8$  to  $2$ , i.e., a multiplicative factor of  $4$ . This means that for each increase of  $x$  by one unit, we multiply  $y$  by  $1/2$  (so that when we increase  $x$  by two units, we multiply  $y$  by  $(1/2)^2 = 1/4$ ). Thus,  $a = 1/2$ . So, we have  $y = C(1/2)^x$ .

Plugging in the point  $(1, 2)$ , we find that  $2 = C(1/2)^1$ , or  $2 = C(1/2)$ , or multiplying both sides by  $2$ ,  $4 = C$ . So, our formula is  $y = 4(1/2)^x$ .

**20. Give a possible formula for the curve shown.**

We want exponential decay asymptoting to the line  $y = 4$ , so we need to do something to our standard  $y = Ca^x$  curve. Example 4 in the text shows us that  $y = S(1 - a^x)$  will give us an asymptote at  $y = S$  (if  $0 < a < 1$  like we always have for exponential decay). So, since we want the asymptote  $y = 4$ , we choose  $S = 4$ . Thus, we have  $y = 4(1 - a^x)$ .

Plugging in the point  $(1, 2)$ , we find that  $2 = 4(1 - a^1)$ , or  $2 = 4(1 - a)$ , or  $2 = 4 - 4a$ . Subtracting  $4$  from both sides, we find  $-2 = -4a$ . Dividing both sides by  $-4$ , we have  $1/2 = a$ . So, the formula is  $y = 4(1 - (1/2)^x)$ .

**39. The median price  $P$  of a home rose from \$ 50000 in 1970 to \$ 100000 in 1990. Let  $t$  be the number of years since 1970.**

**(a) Assume the increase in housing prices has been linear. Give an equation for the line and use it to predict the price at  $t = 10, 30, 40$ .**

The slope of the line would be  $(100000 - 50000)/(1990 - 1970) = 50000/20 = 2500$ . So, the equation would be  $P = 25000t + b$ . Since  $P = 50000$  when  $t = 0$  (in the year 1970), we have  $P = 25000t + 50000$ .

Thus, when  $t = 10$ , we would predict a price of \$ 75000; when  $t = 30$ , we would predict a price of \$ 125000; Thus, when  $t = 40$ , we would predict a price of \$ 150000.

**(b) If instead the housing prices rise exponentially, give an equation and use it to predict the price at the same times.**

The basic equation is  $P = P_0a^t$ . We know that  $P_0 = 50000$  since  $t = 0$  is the year 1970. Thus,  $P = 50000a^t$ . To find  $a$ , we plug in the fact that when  $t = 20$ , we have  $P = 100000$ . So,  $100000 = 50000a^{20}$ . Dividing both sides by  $50000$ , we find  $2 = a^{20}$ , or  $a = \sqrt[20]{2} = 1.0353$ .

So, the formula is  $P = 50000(1.0353)^t$ .

Thus, when  $t = 10$ , we would predict a price of \$ 70711; when  $t = 30$ , we would predict a price of \$ 141421; Thus, when  $t = 40$ , we would predict a price of \$ 200000.

**(c) Sketch the two functions.**

The first is a line and the second is a standard exponential growth shape. Both pass through  $(0, 50000)$  and  $(20, 100000)$ . This means that the exponential growth dips below the line for  $0 < t < 20$  and then rises above it for  $t > 20$  (and becomes much bigger than the line when  $t$  is significantly above 20).

**(d) Which model do you think is more realistic?**

It's tough to say, but values of things tend to grow exponentially, although pure exponential growth is hard to maintain for a long period of time.