HSSI 2007 Math —HW #3 Solution

Problems from §1.5

7. For $\theta = -4\pi/3$, draw the angle using a ray through the origin, and determine whether the $\sin$, $\cos$, and $\tan$ of θ are positive, negative, zero, or undefined.

The ray points into the second quadrant (since $4\pi/3$ is just greater than π, which is 180 degrees, and the minus sign means we measure the angle clockwise rather than counterclockwise).

This means that $\sin\theta > 0$, $\cos\theta < 0$, and $\tan\theta < 0$.

8. For $\theta = 4$, draw the angle using a ray through the origin, and determine whether the $\sin$, $\cos$, and $\tan$ of θ are positive, negative, zero, or undefined.

The ray points into the third quadrant (since 4 is just greater than π, which is 180 degrees, and we measure the angle counterclockwise).

This means that $\sin\theta < 0$, $\cos\theta < 0$, and $\tan\theta > 0$.

10. Given that $\sin(\pi/12) = 0.259$ and $\cos(\pi/5) = 0.809$, what is $\cos(-\pi/5)$?

Since cos is an even function, $\cos(-\pi/5) = \cos(\pi/5) = 0.809$.

11. Given that $\sin(\pi/12) = 0.259$ and $\cos(\pi/5) = 0.809$, what is $\sin(\pi/5)$?

We know that $\cos^2(\pi/5)+\sin^2(\pi/5) = 1$. Since we know that $\cos(\pi/5) = 0.809$, this means that $(0.809)^2 + \sin^2(\pi/5) = 1$, or $\sin^2(\pi/5) = 1 - (0.809)^2 = 0.3455$. Thus, $\sin(\pi/5) = \pm\sqrt{0.3455} = \pm 0.588$. (We get that $\pm$ any time we solve an equation by "taking the square root of both sides", e.g., going from $x^2 = 16$ to $x = \pm 4$).

Which sign is correct for $\sin(\pi/5)$? We know that $\pi/5$ is an angle in the first quadrant (it's less than $\pi/2$, which is 90 degrees), so sin must be positive. Thus, $\sin(\pi/5) = 0.588$.

16. Find the period and amplitude of $w = 8 - 4\sin(2x + \pi)$.

The period comes from what is multiplied by x inside the sin. We saw in class that $\sin(Bx)$ has period $2\pi/B$. Thus, here we have period $2\pi/2 = \pi$.

The amplitude comes from what multiplies sin. In this case, the amplitude is 4 (it doesn't matter that it's negative; that just flips the curve upside down, but the amplitude doesn't change; the amplitude is always a positive number, just by convention).

18. Without a calculator, match the formulas with the graphs.

The function $f(t)$ is $2\cos t$; it's the cos shape, with twice the amplitude.

The function $g(t)$ is $2\cos(t + (\pi/2))$; it's the same as f, but shifted by $\pi/2$ to the left, which means we replace t by $t + (\pi/2)$.

The function $h(t)$ is $2\cos(t - (\pi/2))$; it's the same as f, but shifted by $\pi/2$ to the right, which means we replace t by $t - (\pi/2)$.

24. Find a possible formula for the graph shown.

We'll pick cos rather than sin, since it has that basic shape (it's at a crest at $x = 0$, rather than in the middle of its up-and-down cycle like sin is).

The period is $2\pi/5$ (there are two full cycles in $4\pi/5$). In class we saw that $\sin(Bx)$ or $\cos(Bx)$ has period $2\pi/B$. So, if we want period $2\pi/5$, we take $B = 5$, i.e., consider $\cos(5x)$.

This is almost right – it just needs to have amplitude 2. Thus, one possible formula is $2\cos(5x)$.

27. Find a possible formula for the graph shown.

We'll pick sin rather than cos, since it has that basic shape (it's in the middle of its up-and-down cycle at $x = 0$, rather than at a crest like cos is).

The period is 8π. In class we saw that $\sin(Bx)$ or $\cos(Bx)$ has period $2\pi/B$. So, if we want period 8π, we take $B = 1/4$, since $2\pi/(1/4) = 8\pi$, i.e., consider $\sin(x/4)$.

Next, we fix the amplitude. The amplitude of $\sin(x/4)$ is 1, and we want an amplitude of 2 (one half of $4 - 0$). So, we consider $2\sin(x/4)$.

This is almost right – we just need to shift it up by 2 units. Thus, one possible formula is $2\sin(x/4) + 2$.

Problems from §1.6

1. Which function dominates as $x \to \infty$: $10 \cdot 2^x$ or $72000x^{12}$?

The function $10 \cdot 2^x$ is much larger than $72000x^{12}$ for large values of x: exponential functions a^x (with any $a > 1$) are always eventually larger than any power function x^p, even if p is large.

5. Each of the graphs shown is of a polynomial. The windows are large enough to show global behavior. What is the minimum possible degree of the polynomial? Is the leading coefficient of the polynomial positive or negative?

(I) The minimium possible degree is 3 (it has 2 turning points). The leading coefficient must be negative, since for large values of x, the y-values on the graph are very negative.

(II) The minimium possible degree is 4 (it has 3 turning points). The leading coefficient must be positive, since for large values of x, the y-values on the graph are very positive.

(III) The minimium possible degree is 4 (it has 3 turning points). The leading coefficient must be negative, since for large values of x, the y-values on the graph are very negative.

(IV) The minimium possible degree is 5 (it has 4 turning points). The leading coefficient must be negative, since for large values of x, the y-values on the graph are very negative.

(V) The minimium possible degree is 5 (it has 4 turning points). The leading coefficient must be positive, since for large values of x, the y-values on the graph are very positive.

6. For each function, determine what f approaches as $x \to -\infty$ and as $x \to +\infty$.

(a) $f(x) = 17 + 5x^2 - 12x^3 - 5x^4$

As $x \to +\infty$, $f(x)$ is dominated by the highest-degree term $-5x^4$, and thus f is approaching $-\infty$.

As $x \to -\infty$, $f(x)$ is dominated by the highest-degree term $-5x^4$, and thus f is approaching $-\infty$.

(b) $f(x) = \frac{3x^2-5x+2}{2x^2-8}$

As $x \to +\infty$, the numerator of $f(x)$ is dominated by the highest-degree term $3x^2$, and the denominator of $f(x)$ is dominated by the highest-degree term $2x^2$, so the entire fraction $f(x)$ acts like $(3x^2)/(2x^2)$, or $3/2$. So, f is approaching $3/2$.

As $x \to -\infty$, the numerator of $f(x)$ is dominated by the highest-degree term $3x^2$, and the denominator of $f(x)$ is dominated by the highest-degree term $2x^2$, so the entire fraction $f(x)$ acts like $(3x^2)/(2x^2)$, or $3/2$. So, f is approaching $3/2$.

(c) $f(x) = e^x$

We've seen the graph of e^x, so keep that picture in mind.

We know that as $x \to +\infty$, the graph of e^x shoots up to $+\infty$.

Similarly, as $x \to -\infty$, the graph levels off and approaches the x-axis as a horizontal asymptote. In the language of this problem, we'd say $f(x) \to 0$ as $x \to -\infty$.

12. Find a cubic polynomial for the graph shown.

Notice that we want the function to equal zero when $x = -2$, $x = 1$, and $x = 5$. There's a simple formula that does this: $(x+2)(x-1)(x-5)$. Notice that when you plug in any of those x values (-2, 1, or 5), you get zero, like you want. Plus, this is a cubic function (a product of three linear factors; you could do FOIL to expand it out so it looked like a standard cubic polynomial).

However, when we plug in $x = 0$ into $(x+2)(x-1)(x-5)$, we get $(2)(-1)(-5) = 10$, and the picture shows that when $x = 0$, we want to get $y = 2$. To get this, we can just divide our formula by 5: $f(x) = \frac{1}{5}(x+2)(x-1)(x-5)$. It still equals zero where we want, and now $f(0) = 2$ like we want as well.

14. Which of the functions I–III meet each of the following descriptions, where (I) is $f(x) = \frac{x-1}{x^2+1}$, (II) is $f(x) = \frac{x^2-1}{x^2+1}$, and (III) is $f(x) = \frac{x^2+1}{x^2-1}$.

(a) Horizontal asymptote of $y = 1$

This means that when $x \to +\infty$, we have $f(x) \to 1$. This is true for (II) and (III), since in each case, for very large positive values of x, the numerator acts like x^2 (the highest degree term), and the denominator acts like x^2 (same reason), so the fraction behaves like x^2/x^2, which is 1, so $f(x) \to 1$.

(On the other hand, for I, for vary large positive values of x, the numerator behaves like x, the denominator behaves like x^2, so the fraction behaves like $x/x^2 = 1/x$, which goes to 0, not 1, for large values of x.)

(b) The x-axis is a horizontal asymptote

This is true for (I). See my reasoning at the end of part (a); having the x-axis as a horizontal asymptote is another way of saying that as $x \to +\infty$, we have $f(x) \to 0$.

(c) Symmetric about the y-axis

This is true for (II) and (III), since all powers of x are even, so the whole function is even, and thus has this symmetry. Algebraically, it is pretty easy to check that if you plug in $-x$ for x, you get the same answer, i.e., $f(-x) = f(x)$, which is the definition of an even function.

(d) An odd function

This is not true for any of the functions. Clearly it is not true for (II) or (III), since they are even, not odd. Thus, only (I) is a candidate. We need $f(-x) = -f(x)$ in order for the function to be odd. However, when we plug in $-x$ for x, the denominator of (I) stays the same, but the numerator becomes something not easily related to what it was before: it was $x - 1$, and it becomes $-x - 1$, which is $-(x + 1)$, not $-(x - 1)$.

If that's confusing, draw the graph, and notice that it doesn't have that 180-degree rotation symmetry that odd functions must have.

(e) Vertical asymptotes at $x = \pm 1$

This is true for (III), since the denominator vanishes at $x = \pm 1$.

17. According to Car and Driver, an Alfa Romeo going at 70 mph requires 177 ft to stop. Assuming that the stopping distance is proportional to the square of the velocity, find the stopping distances requires by an Alfa Romeo going at 35 mph and 140 mph.

Since the distance is proportional to the square of the velocity, we have $D = kv^2$ (the k is an unknown constant, as we always have when we say "proportional to").

We can use the information given by the problem to find k, since it says that when $v = 70$, we have $D = 177$. Thus, $177 = k(70)^2$, or $k = \frac{177}{(70)^2} = 0.036122$.

Now we can answer the given questions: when $v = 35$, the distance is $0.036122(35)^2 = 44.25$ feet, and when $v = 140$, the distance is $0.036122(140)^2 = 708$ feet.

24. Given the tables of three functions, which is exponential, which is quadratic,

and which is cubic?

The middle function, $g(x)$, is the exponential function. How do I know? I looked for the key feature that is the hallmark of all exponential functions, and only exponential functions: when I increase x by some fixed amount, I always multiply y by the same number. Here, when I increase x by 0.5, I multiply y by 1.2. You can check that 3.74/3.12, 4.49/3.74, 5.39/4.49, 6.47/5.39, and 7.76/6.47 all equal 1.2, at least within rounding error.

Now, the question becomes, is $k(x) = ax^2$, or $k(x) = ax^3$? Notice that we are given the values of k at $x = 0.6$, and also at $x = 1.8$, which is 3 times larger than 0.6. If $k(x) = ax^2$, then $k(1.8)$ would be 9 times bigger than $k(0.6)$, since $9 = 3^2$. If $k(x) = ax^3$, then $k(1.8)$ would be 27 times bigger than $k(0.6)$, since $27 = 3^3$. If you look at the table, you'll notice that $k(0.6)$ is 3.24 and $k(1.8)$ is 9 times larger, i.e., 29.19. So, k is the quadratic function.

By process of elimination, f is the cubic function. If you want to check it, pick any two x values, say $x = 9.0$ and $x = 10.8$. Notice that 10.8 is 1.2 times 9.0. Now look at the corresponding y values, 7.29 and 12.60. Notice that 12.60 is 1.728 times 7.29 and $1.728 = (1.2)^3$, verifying the cubic nature of f.