

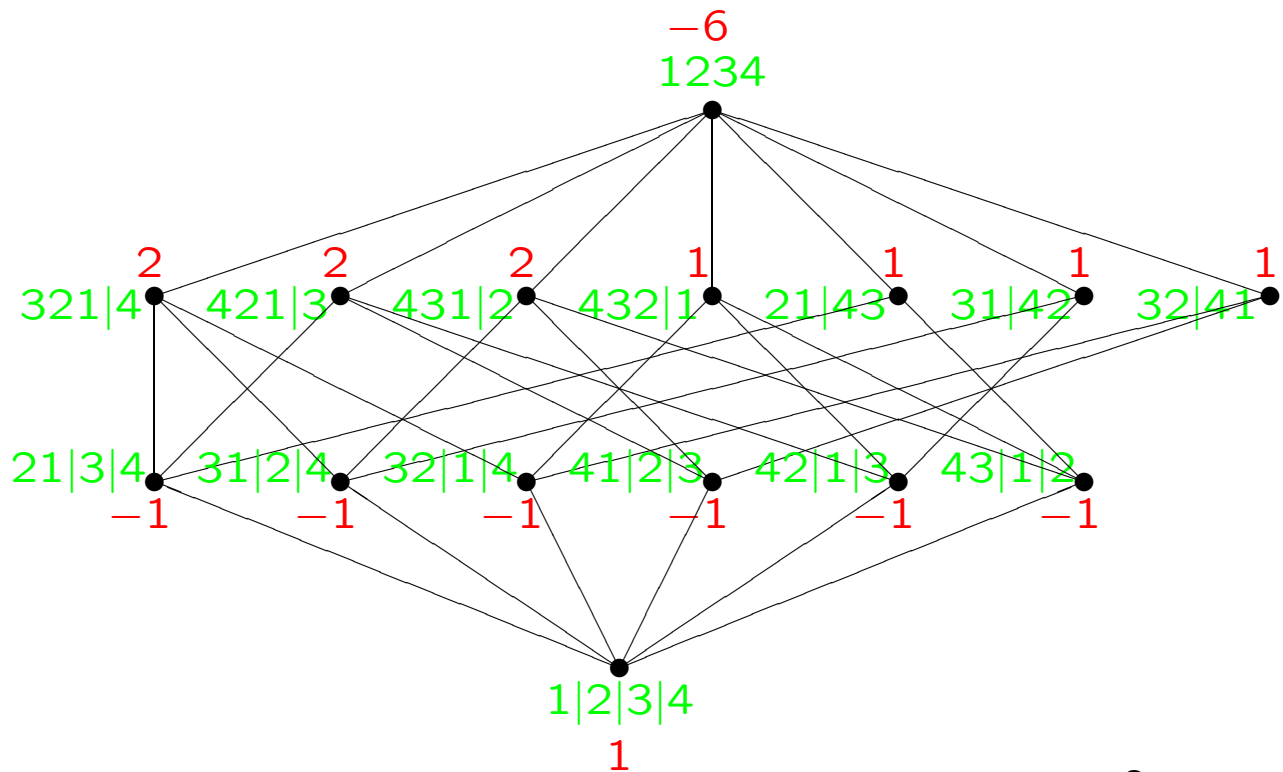
Lecture #3: Topology of Posets

Order complexes and the Cohen-Macaulay property

10:30 – 11:30 a.m.
August 16, 1996

The Set Partition Lattice

For the set partition lattice Π_n , the Möbius number $\mu_n(\hat{0}, \hat{1})$ is -1 for $n = 2$, 2 for $n = 3$, and -6 for $n = 4$. This lecture provides a topological explanation of why $\mu_n(\hat{0}, \hat{1}) = (-1)^{n-1}(n-1)!$, and why $\mu(\hat{0}, x)$ has the same sign as $(-1)^{\text{rank}(x)}$.



Properties of Familiar Posets

Posets: (distributive, modular, **semimodular**)

B_n = subsets of the set $[n] = [0, 1]^n$,

$[0, \lambda_1] \times \cdots \times [0, \lambda_\ell] = J([\lambda_1] + \cdots + [\lambda_\ell])$,

$B_n(q)$ = subspaces of the vector space $(\mathbf{F}_q)^n$,

normal subgroups of a finite group,

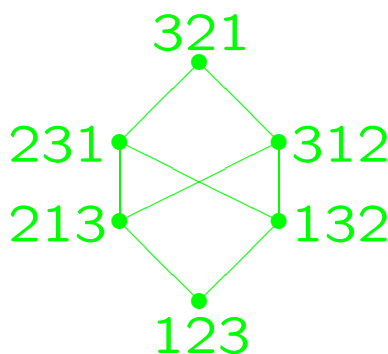
Π_n = **partitions of the set $[n]$** ,

Bruhat order on the symmetric group S_n .

Properties:

Each is a finite, graded poset $\hat{P} = P \cup \{\hat{0}, \hat{1}\}$ whose Möbius function alternates in sign (that is, $(-1)^{\text{rank}(y) - \text{rank}(x)} \mu(x, y) \geq 0$), and whose order complex $\Delta(P)$ has the homology of a wedge of spheres of dimension $\dim(\Delta(P))$.

Bruhat order on S_n
is not a lattice.



Order Complexes and Face Lattices

Definition: Let $\hat{P} = P \cup \{\hat{0}, \hat{1}\}$ be a finite poset. The **order complex** $\Delta(P)$ is the simplicial complex of nonempty chains in P . If \hat{P} is graded of rank n , then $\Delta(P)$ is pure of dimension $n - 2$.

Definition: The face poset $F(\Gamma)$ of a finite complex Γ is the poset of its simplices ordered by inclusion. Notice $\Delta(F(\Gamma)) = \text{sd}(\Gamma)$. If Γ is pure of dimension $n - 2$, then the **face lattice** $\widehat{F(\Gamma)}$ is graded of rank n .

$$\begin{array}{ccc}
 \hat{P} & \longrightarrow & \Delta(P) \text{ order complex} \\
 \text{face lattice } \widehat{F(\Gamma)} & \longleftarrow & \Gamma
 \end{array}$$

Hall's Theorem: If $\hat{P} = P \cup \{\hat{0}, \hat{1}\}$ is finite and graded, then

$$\mu_{\hat{P}}(\hat{0}, \hat{1}) = \tilde{\chi}(\Delta(P)).$$

where $\tilde{\chi}(\Delta) = -1 + c_0 - c_1 + \dots$ if Δ has c_i i -simplices, $i \geq 0$.

The Cohen-Macaulay property

Definition: A finite, graded poset \hat{P} is **Cohen-Macaulay** (over \mathbf{Q}) if for every $x < y$ in \hat{P} , the order complex $\Delta(x, y)$ of the open interval (x, y) satisfies

$$\tilde{H}_i(\Delta(x, y), \mathbf{Q}) = 0, \text{ if } i < \dim(\Delta(x, y)).$$

Theorem: If \hat{P} is Cohen-Macaulay, then the Möbius function of \hat{P} alternates in sign.

Proof: Since $\tilde{H}_i(\Delta(x, y), \mathbf{Q}) = 0$ for $i < d = \dim(\Delta(x, y))$, the Euler-Poincaré formula gives

$$\tilde{\chi}(\Delta(x, y)) = (-1)^d \dim_{\mathbf{Q}} \tilde{H}_d(\Delta(x, y), \mathbf{Q}).$$

On the other hand, Hall's theorem applied to $[x, y]$ gives

$$\mu_{\hat{P}}(x, y) = \tilde{\chi}(\Delta(x, y)).$$

Hence

$$\mu_{\hat{P}}(x, y) = (-1)^d \dim_{\mathbf{Q}} \tilde{H}_d(\Delta(x, y), \mathbf{Q}),$$

where $d = \text{rank}(y) - \text{rank}(x) - 2$.

Rank Selection

Theorem: Let $\hat{P} = P \cup \{\hat{0}, \hat{1}\}$ be a finite, graded poset of rank n . For $S \subseteq [n - 1]$, let P_S be the subposet of elements of P whose ranks are in S . If \hat{P} is Cohen-Macaulay, then the **rank-selected subposet** $\hat{P}_S = P_S \cup \{\hat{0}, \hat{1}\}$ is Cohen-Macaulay.

Corollary Let $\alpha(S)$ be the number of maximal chains in \hat{P}_S . Define

$$\beta(S) = \sum_{T \subseteq S} (-1)^{|S-T|} \alpha(T).$$

If \hat{P} is Cohen-Macaulay, then $\Delta(P_S)$ has the homology of a wedge of $\beta(S)$ spheres of dimension $|S| - 1$.

Example: For the Boolean algebra B_n , $\alpha(S)$ is the number of permutations of $[n]$ whose descent set is contained in S . So $\beta(S)$ is the number of permutations of $[n]$ with descent set equal to S .

Shellable Simplicial Complexes

Definition: Let $\text{bd}(\sigma)$ denote the boundary complex of the simplex σ . A finite, pure, d -dimensional complex Δ is **shellable** if its d -simplices can be ordered $\sigma_1, \sigma_2, \dots, \sigma_t$ so that

$$(\text{bd}(\sigma_1) \cup \dots \cup \text{bd}(\sigma_{k-1})) \cap \text{bd}(\sigma_k)$$

is a pure $(d - 1)$ -dimensional complex for $2 \leq k \leq t$.

Note: A shellable d -dimensional complex is $(d - 1)$ -connected, hence is homotopy equivalent to a wedge of d -spheres.

Theorem: Let $\hat{P} = P \cup \{\hat{0}, \hat{1}\}$ be a finite, graded poset of rank n . If $\Delta(P)$ is shellable, then \hat{P} is Cohen-Macaulay.

Shelling orders can be obtained from certain labelings of the edges of the Hasse diagram.

Shelling of an abstract simplicial complex

The subcomplex

$$(\text{bd}(\sigma_1) \cup \cdots \cup \text{bd}(\sigma_{k-1})) \cap \text{bd}(\sigma_k)$$

does not equal

$$(\sigma_1 \cup \cdots \cup \sigma_{k-1}) \cap \sigma_k.$$

Example: Let $\sigma_1 = \{1, 2, 3\}$, $\sigma_2 = \{1, 2, 4\}$, $\sigma_3 = \{1, 3, 4\}$, and $\sigma_4 = \{2, 3, 4\}$ be in Δ . Since $\text{bd}(\{i, j, k\}) = \{\{i\}, \{j\}, \{k\}, \{i, j\}, \{i, k\}, \{j, k\}\}$,

$$(\text{bd}(\sigma_1) \cup \text{bd}(\sigma_2) \cup \text{bd}(\sigma_3)) \cap \text{bd}(\sigma_4) = \text{bd}(\sigma_4),$$

but

$$(\sigma_1 \cup \sigma_2 \cup \sigma_3) \cap \sigma_4 = \sigma_4.$$

Note: If Δ is shellable, then Δ is homotopy equivalent to a wedge of β d -spheres, where $\beta = |\{k \mid \text{bd}(\sigma_k) \subseteq \text{bd}(\sigma_1) \cup \cdots \cup \text{bd}(\sigma_{k-1})\}|$ and $\sigma_1, \sigma_2, \dots, \sigma_t$ is any shelling.

Semimodular Lattices

Definition: An edge labeling λ of the Hasse diagram of \hat{P} is an **EL-labeling** if every interval $[x, y]$ in \hat{P}

(i) has a unique maximal chain $x_0 < \cdots < x_k$ such that $\lambda(x_0 < x_1) \leq \cdots \leq \lambda(x_{k-1} < x_k)$, and
(ii) the label sequence for every other maximal chain in $[x, y]$ is lexicographically later than the label sequence for this unique (weakly) rising chain.

Theorem: If L is a finite semimodular lattice, then its Hasse diagram has an EL-labeling, and $\beta(S)$ is the number of maximal chains in L whose label sequences have descent set S .

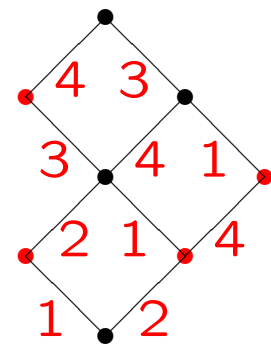
Idea of the Proof: Fix any linear ordering of the join-irreducibles e_i in L that extends their partial ordering. If y covers x , let

$$\lambda(x < y) = \min\{i \mid x \vee e_i = y\}.$$

Distributive Lattices

Corollary: If $L = J(P)$ is a finite distributive lattice and $P \rightarrow [n]$ is a fixed order-preserving bijection, then $\beta(S)$ is the number of linear extensions of P that have descent set S .

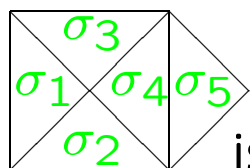
Example: Let $P = \begin{array}{c} 3 \\ | \\ 1 \end{array} \begin{array}{c} 4 \\ | \\ 2 \end{array}$ so $L =$
 The linear extensions, maximal
 simplices for $S = \{1, 2, 3\}$, and
 maximal simplices for $S = \{1, 3\}$:



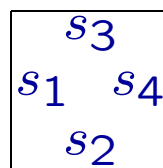
- 1234: $\sigma_1 = \{1 \subset 12 \subset 123\}$, $s_1 = \{1 \subset 123\}$;
- 1243: $\sigma_2 = \{1 \subset 12 \subset 124\}$, $s_2 = \{1 \subset 124\}$;
- 2134: $\sigma_3 = \{2 \subset 12 \subset 123\}$, $s_3 = \{2 \subset 123\}$;
- 2143: $\sigma_4 = \{2 \subset 12 \subset 124\}$, $s_4 = \{2 \subset 124\}$;
- 2413: $\sigma_5 = \{2 \subset 24 \subset 124\}$.

No linear extensions with descent set $\{1, 2, 3\}$, so

Exactly one with descent set $\{1, 3\}$,



is contractible.



so is a circle.

Stanley's Shelling of the Set Partition Lattice

Theorem: Order the join-irreducibles of Π_n as follows:

$$21 < 31 < 32 < 41 < 42 < 43 < \dots,$$

where ji is the partition with blocks $\{i, j\}$ and $\{k\}$, all $k \notin \{i, j\}$. Then the maximal chains whose label sequences have descent set $[n-1]$ are those of the form

$$\hat{0} < x_{n-1} < x_{n-1} \vee x_{n-2} < \dots < x_{n-1} \vee \dots \vee x_1 < \hat{1},$$

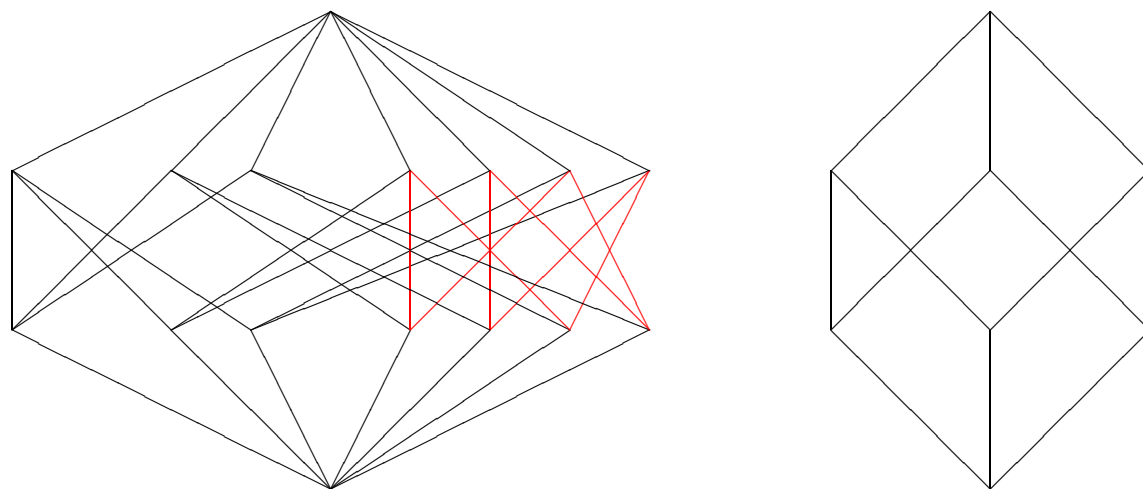
where $x_1 = 21$, $x_2 \in \{31, 32\}$, $x_3 \in \{41, 42, 43\}$, and so on.

Corollary: The Möbius number $\mu(\hat{0}, \hat{1})$ of the set partition lattice Π_n is $(-1)^{n-1}(n-1)!$.

Order Analogues and Betti Polynomials

Since $B_n(q)$ is a modular lattice, the order complex of subspaces whose dimensions are in S is homotopy equivalent to a wedge of $\beta(S; q)$ spheres of dimension $|S| - 1$. This polynomial is the sum, over permutations π of $[n]$ with descent set S , of q raised to the inversion number of π . A topological explanation of this fact uses Knuth's order preserving surjection of $B_n(q)$ onto B_n .

Example: The Betti polynomial $\beta(S; q)$ for the lattice of subspaces evaluates at $q = 1$ to the Betti number $\beta(S)$ for the lattice of subsets.



References

Anders Björner, et al, An Introduction to Cohen-Macaulay Partially Ordered Sets in Ordered Sets.

Anders Björner and Michelle Wachs, Bruhat order of Coxeter groups and shellability.

Anders Björner and Michelle Wachs, Shellable nonpure complexes and posets.

Lynne Butler, Order analogues and Betti polynomials.

James Munkres, Topological Results in Combinatorics.

Richard Stanley, Enumerative Combinatorics.

Richard Stanley, Finite lattices and Jordan-Hölder sets.

Related Talks at the October Workshop

Anders Björner, “Subspace arrangements over finite fields: enumeration and cohomology” .

Lynne Butler, “Enumeratively identical lattices” .

Philip Hanlon, “Inclusion-Exclusion and Euler characteristics” .

Victor Reiner, “Resolutions, simplicial complexes, and shellability” .

Sheila Sundaram, “Homology representations of posets: techniques and results” .

John Shareshian, “Nonpure shellability of subgroup lattices” .

Michelle Wachs, “Geometrically constructed homology bases for real hyperplane intersection lattices” .