Lecture #2: Partially Ordered Sets

Lattices
and
Möbius Functions

10:30 – 11:30 a.m.
August 14, 1996

Examples of Partially Ordered Sets

Definition: A poset is a set P together with a binary relation $\leq$ that is
reflexive ($x \leq x$),
antisymmetric ($x \leq y$ and $y \leq x \Rightarrow x = y$),
transitive ($x \leq y$ and $y \leq z \Rightarrow x \leq z$).

Basic examples: $[m, n]$, B_n, D_n, and Π_n are
$\{m, m+1, \ldots, n\}$ ordered by $\leq$,
subsets of $[n]$ ordered by inclusion,
divisors of n ordered by divisability,
partitions of $[n]$ ordered by refinement.

More examples: Order by inclusion the
Ferrers diagrams of partitions,
normal subgroups of a (finite) group,
subspaces the vector space $(\mathbf{F}_q)^n$,
faces of a polytope.

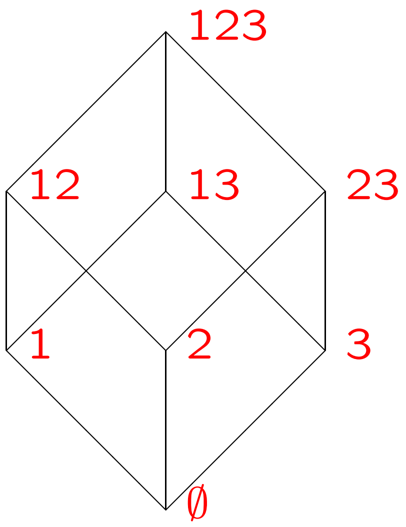
Isomorphic Posets and Hasse diagrams

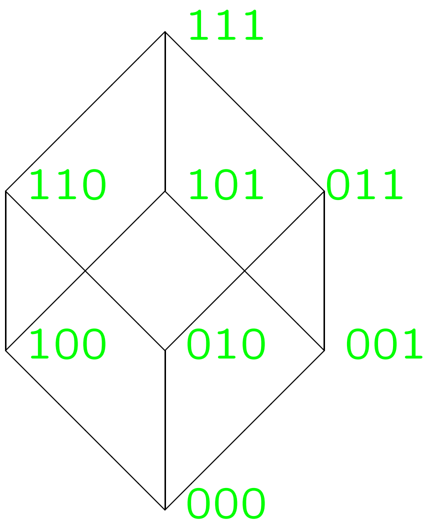
Definition: Posets P and Q are isomorphic if there is a bijection $\varphi : P \to Q$ such that $x \leq y \Leftrightarrow \varphi(x) \leq \varphi(y)$.

Example: $D_n \cong [0, m_1] \times [0, m_2] \times \cdots \times [0, m_k]$ if $n = p_1^{m_1} p_2^{m_2} \cdots p_k^{m_k}$ for primes p_i. $B_n \cong [0, 1]^n$.

Definition: The Hasse diagram of P is the directed graph with vertex set P and an edge from x up to y if y covers x ($x < y$ and there is no z such that $x < z < y$).

Example: Hasse diagrams for B_3 and $[0, 1]^3$.



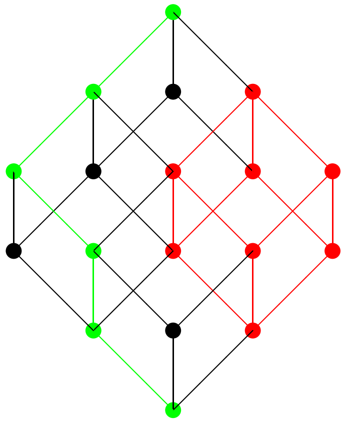


Chains and Intervals

Definition: A chain in P is a totally ordered subset of P. The chain $x_0 < x_1 < \cdots < x_\ell$ has length ℓ. A finite poset is graded of rank n if every maximal chain has length n.

Each poset mentioned above has one minimal element $\widehat{0}$ and finite, graded intervals $[x, z] = \{y \,|\, x \leq y \leq z\}$. Each has finitely many elements of each rank, where $\text{rank}(\widehat{0}) = 0$ and $\text{rank}(y) = \text{rank}(x) + 1$ if y covers x.

Example A maximal chain and interval in D_{180}.



Lattices

Definition: A poset L is a lattice if every pair of elements $x, y \in L$ has a greatest lower bound $x \wedge y$ (x meet y) and a least upper bound $x \vee y$ (x join y).

Examples: In B_n, intersection and union. In D_n, greatest common divisor and least common multiple. In both, $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$, so these lattices are distributive.

Example: The lattice of subspaces of $(F_q)^n$ is not distributive for $n \geq 2$, but x and y cover $x \wedge y$ if and only if $x \vee y$ covers x and y. So this lattice is modular.

Example: The lattice Π_n of partitions of $[n]$ is not modular for $n \geq 4$, because $12|34 \vee 13|24 = 1234$, but $12|34 \wedge 13|24 = 1|2|3|4$. However, since $x \vee y$ covers x and y if x and y cover $x \wedge y$, this lattice is semimodular.

Distributive Lattices: Birkhoff's Theorem

Definition: An element in a lattice L is join-irreducible if it is not the join of smaller elements.

Example: In B_n, singleton subsets. In D_n, prime powers. In Young's lattice of partitions, a partition is join-irreducible if and only if all its parts are equal.

Definition: A subset I of a poset P is an ideal if it is closed under $\leq$ ($x \leq y \in I \Rightarrow x \in I$).

The set $J(P)$ of finite ideals of P, ordered by inclusion, is a distributive lattice.

Example: $[0, \lambda_1] \times [0, \lambda_2] \times \cdots \times [0, \lambda_\ell]$ is isomorphic to $J([\lambda_1] + [\lambda_2] + \cdots + [\lambda_\ell])$.

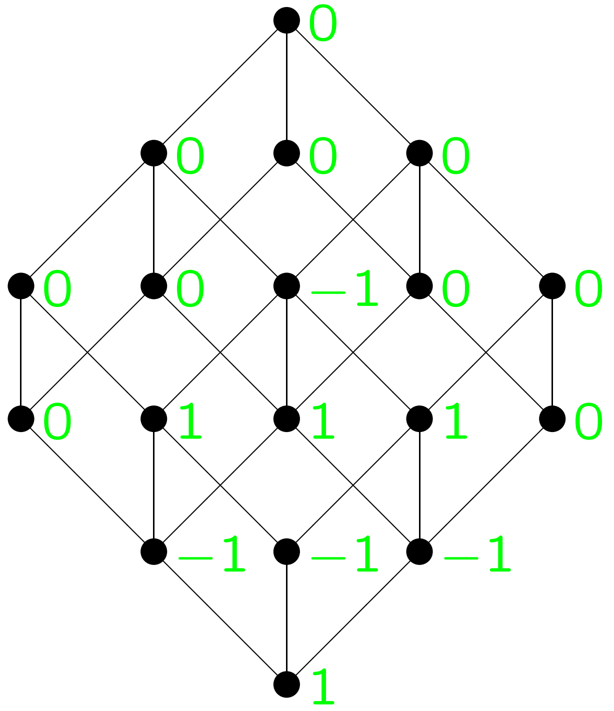
Theorem: Let L be a distributive lattice with $\hat{0}$ and finite intervals, and let P be its nonzero join-irreducibles. Then $L \cong J(P)$.

The Möbius Function

Definition: Let P be a poset whose intervals are finite and $Int(P)$ be the set of its intervals. The Möbius function $\mu : Int(P) \to \mathbf{Z}$ is defined by

$$\mu(x,x) = 1, \text{ for } x \in P$$
$$\mu(x,z) = -\sum_{x \le y < z} \mu(x,y), \text{ for } x < z \text{ in } P.$$

Example: Beside $z \in D_{180}$ we record $\mu(\widehat{0}, z)$.



Properties of the Möbius Function

Theorem: If $(x, x') \leq (z, z')$ in $P \times P'$, then

$$\mu_{P\times P'}((x, x'), (z, z')) = \mu_P(x, z)\mu_{P'}(x', z').$$

Proof: Let $(x, x') \leq (z, z')$.

$$\sum_{(x,x')\leq(y,y')\leq(z,z')} \mu_P(x, y)\mu_{P'}(x', y')$$
$$= \left(\sum_{x\leq y\leq z} \mu_P(x, y)\right)\left(\sum_{x'\leq y'\leq z'} \mu_{P'}(x', y')\right)$$
$$= \delta_{x,z}\delta_{x',z'} = \delta_{(x,x'),(z,z')}$$

Example: If $L = [0, \lambda_1] \times [0, \lambda_2] \times \cdots [0, \lambda_\ell]$, then

$$\mu_L(m_1, m_2 \ldots , m_\ell) = \begin{cases} (-1)^{\sum m_i} & m_i = 0 \text{ or } 1 \\ 0 & \text{otherwise.} \end{cases}$$

Corollary: For $m \in D_n$, $\mu_{D_n}(\hat{0}, m) = \mu(m)$, where μ is the Möbius function from number theory.

Möbius Inversion and Inclusion-Exclusion

Theorem: Let P be a poset whose principal order ideals $\{y \,|\, y \leq x\}$ are finite. If α and β map elements of P to elements of an abelian group, then

$$\alpha(y) = \sum_{x \leq y} \beta(x), \text{ for all } y$$

if and only if

$$\beta(y) = \sum_{x \leq y} \mu_P(x,y)\alpha(x), \text{ for all } y.$$

Examples: For D_n, this is Möbius Inversion from number theory. For B_n, this is Inclusion-Exclusion, because $\mu_{B_n}(T,S) = (-1)^{|S-T|}$.

How are the values of the Möbius function computed and for which posets does it have geometric or topological significance?

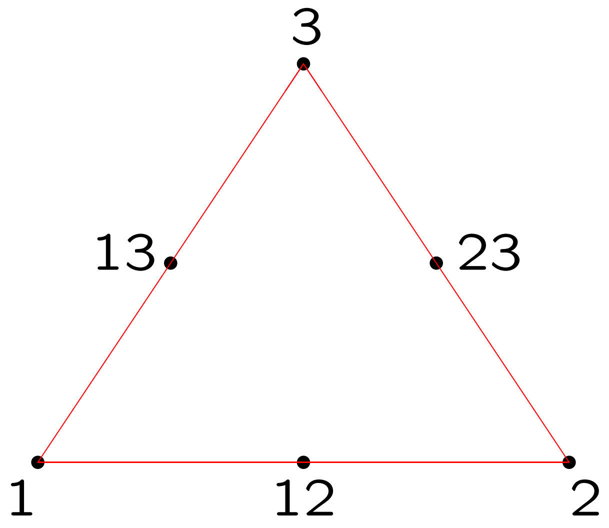
Philip Hall's Theorem

Theorem: Let $\widehat{P} = P \cup \{\hat{0}, \hat{1}\}$ be a finite, graded poset and let $\Delta(P)$ be the simplicial complex of chains in P. Then

$$\mu_{\widehat{P}}(\hat{0}, \hat{1}) = \tilde{\chi}(\Delta(P)),$$

where $\tilde{\chi}(\Delta) = -1 + c_0 - c_1 + \cdots$ is the reduced Euler characteristic of a simplicial complex Δ that has c_i i-simplices, $i \geq 1$.

Example: If $\widehat{P} = B_3$, then the order complex $\Delta(P)$ is



and $\tilde{\chi}(\Delta(P)) = -1 + 6 - 6 = -1$.

Rota's Cross-Cut Theorem

Theorem: Let L be a finite lattice and let A be a subset of L such that

$\widehat{0} \notin A$

$\widehat{0} \neq x \in L \Rightarrow a \leq x$ for some $a \in A$.

Then

$$\mu(\widehat{0}, \widehat{1}) = \sum_k (-1)^k A_k,$$

where A_k is the number of k-subsets of A whose join is $\widehat{1}$.

Example: Use Rota's cross-cut theorem to deduce the following result of Matveev's from Alexander duality: If $\widehat{P} = P \cup \{\widehat{0}, \widehat{1}\}$ is finite and G is the graph whose vertices are elements of P and whose (undirected) edges are (unordered) pairs of incomparable elements in P, then

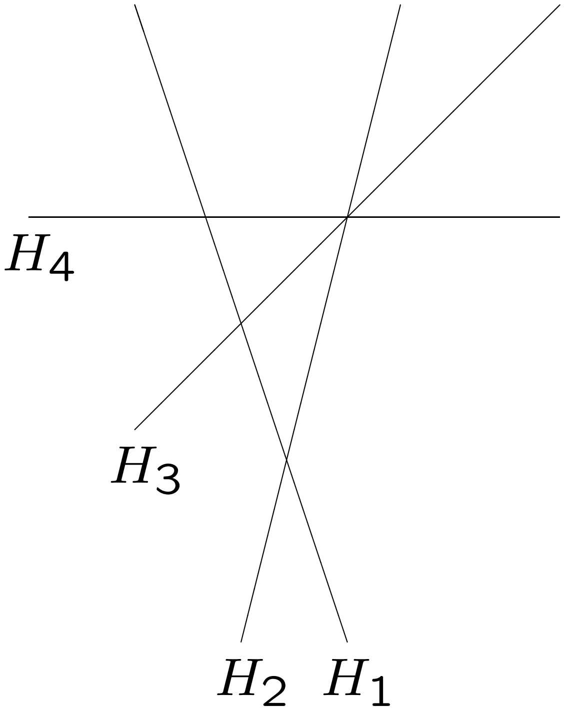
$$\mu(\widehat{0}, \widehat{1}) = \sum_k (-1)^k A_k,$$

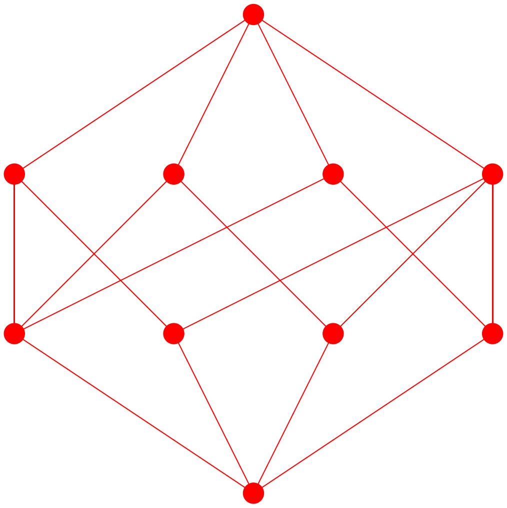
where A_k is the number of k-subsets of edges that cover the vertices in G.

Hyperplane Arrangements

Definition: Let $H_1, H_2, \ldots, H_\ell$ be hyperplanes in $\mathbf{R}^d$ such that $H_1 \cap H_2 \cap \cdots \cap H_n = \emptyset$. The intersection lattice of this arrangement is the set of distinct intersections $H_{i_1} \cap H_{i_2} \cap \cdots \cap H_{i_k}$, ordered by reverse inclusion. So $\widehat{0} = \mathbf{R}^d$ and $\widehat{1} = \emptyset$.

Example: The arrangement of lines shown at left has the intersection lattice shown at right.



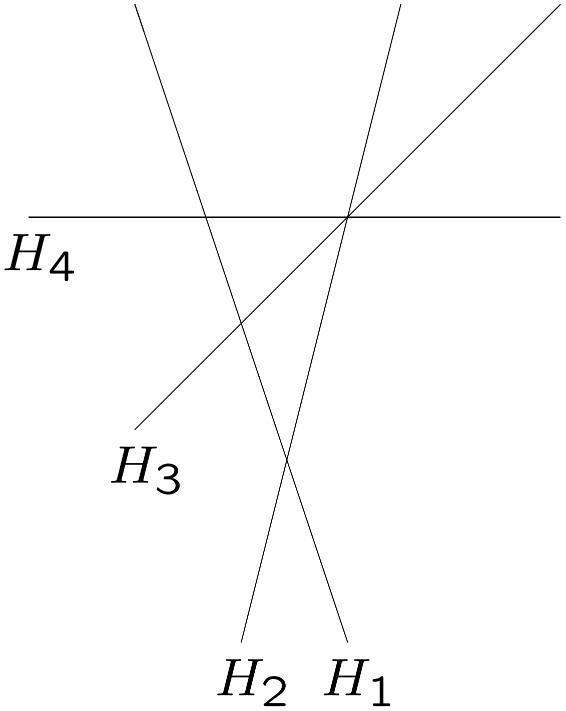


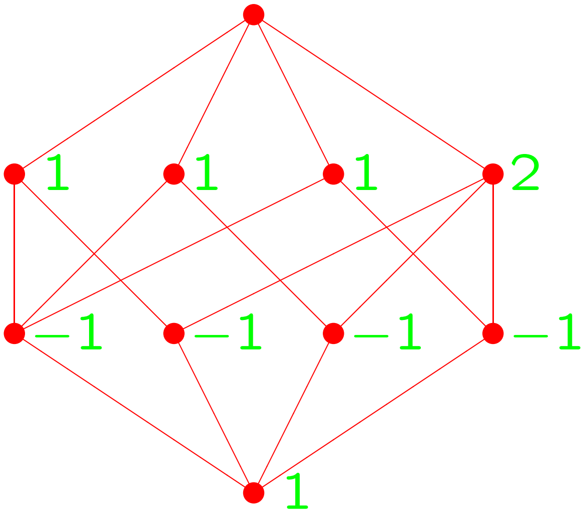
Zaslavsky's Result

Theorem: The number R of regions and the number B of bounded regions cut out by a hyperplane arrangement are:

$$R = \sum_{x \neq \hat{1} \in L} |\mu_L(\hat{0}, x)| \qquad B = |\sum_{x \neq \hat{1} \in L} \mu_L(\hat{0}, x)|.$$

Example: There are 10 regions of which 2 are bounded.





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