MSRI Joint Introductory Workshop

Enumeration
and
Partially Ordered Sets

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Lecture #1: Basic Enumeration

Generating Functions
and
Inclusion-Exclusion

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Ordinary Generating Functions

Definition: A partition of a number n is a non-increasing sequence of positive integers, called parts, whose sum is n. Let $p(n)$ denote the number of partitions of n.

Example: The partitions of 5 are

$$5,\ 41,\ 32,\ 311,\ 221,\ 2111,\ 11111,$$

so $p(5) = 7$.

Euler knew a simple expression for the formal power series

$$\sum_{n\geq 0} p(n)x^n = 1 + x + 2x^2 + 3x^3 + 5x^4 + 7x^5 + \cdots$$

Ramanujan found a beautiful but complicated formula for $p(n)$ (an infinite series that involves derivatives of hyperbolic functions).

The Hardy-Ramanujan-Rademacher Formula for $p(n)$

Theorem: The number $p(n)$ of partitions of n is

$$\frac{1}{\pi\sqrt{2}}\sum_{k=1}^{\infty} A_k(n)k^{\frac{1}{2}}\left[\frac{d}{dx}\frac{\sinh(\frac{\pi}{k}(\frac{2}{3}(x-\frac{1}{24}))^{\frac{1}{2}})}{(x-\frac{1}{24})^{\frac{1}{2}}}\right]_{x=n}$$

where

$$A_k(n) = \sum_{\substack{h \bmod k \\ (h,k)=1}} \omega_{h,k}e^{-2\pi inh/k},$$

and the 24th root of unity

$$\omega_{h,k} = \exp(\pi i s(h,k))$$

is defined by

$$s(h,k) = \sum_{\mu=1}^{k-1}\left(\frac{\mu}{k}-\lfloor\frac{\mu}{k}\rfloor-\frac{1}{2}\right)\left(\frac{h\mu}{k}-\lfloor\frac{h\mu}{k}\rfloor-\frac{1}{2}\right).$$

Euler's Generating Function for Partitions

Theorem: The ordinary generating function for the numbers $p(n)$ that count partitions is

$$\sum_{n\geq 0} p(n)x^n = \prod_{i\geq 1}(1-x^i)^{-1}.$$

Proof: A finite multiset $\{1^{m_1}, 2^{m_2}, 3^{m_3}, \ldots\}$ is the multiset of parts of a partition λ of n if and only if $\sum_{i\geq 1} im_i = n$. Write the infinite product $(1+x+x^2+\cdots)(1+x^2+x^4+\cdots)(1+x^3+x^6+\cdots)\cdots$ as a sum of monomials $x^{m_1}x^{2m_2}x^{3m_3}\cdots$ over finitely nonzero sequences $(m_1, m_2, m_3, \ldots)$ of nonnegative integers. Thereby observe that the coefficient of x^n in $\prod_{i\geq 1}(1-x^i)^{-1}$ is the number of sequences with $\sum_{i\geq 1} im_i = n$.

Example: The monomial $x^2 1 x^3 = x^{1\cdot 2}x^{2\cdot 0}x^{3\cdot 1}$ for the sequence $(2, 0, 1)$ corresponds to the partition 311 with $m_1 = 2$ and $m_3 = 1$.

An Application of Generating Functions

Example: The partitions of 5 into distinct parts are 5, 41, and 32. The partitions of 5 into odd parts are 5, 311, and 11111.

Corollary: The number of partitions of n into distinct parts equals the number of partitions of n into odd parts.

Proof: The ordinary generating function for partitions with distinct parts is

$$\sum_{n \geq 0} p_{\text{dist}}(n)x^n = (1+x)(1+x^2)(1+x^3)\cdots.$$

Write each factor $(1+x^i)$ as $\frac{(1-x^{2i})}{(1-x^i)}$. Cancel the numerator with all of the factors $(1-x^{2i})$ in the denominator, leaving

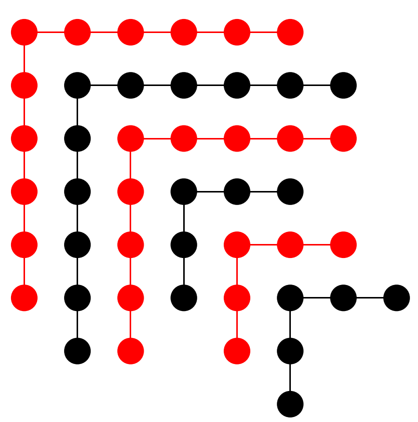
$$\frac{1}{(1-x)(1-x^3)(1-x^5)\cdots} = \sum_{n \geq 0} p_{\text{odd}}(n)x^n,$$

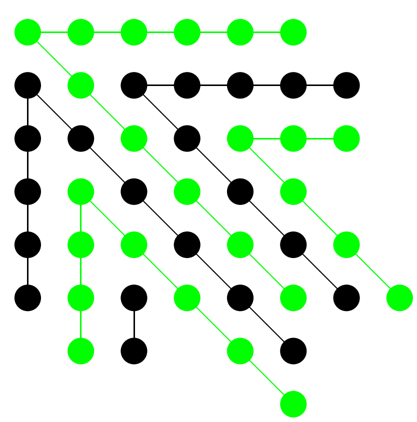
the ordinary generating function for partitions with odd parts.

Sylvester's Bijection

Theorem: The number of partitions of n into odd parts of which exactly k are distinct equals the number of partitions of n into distinct parts in which exactly k sequences of consecutive integers occur.

Idea of the Proof: Picture the odd parts of 11 11 9 5 5 5 as the nested hooks shown at left; read off distinct parts from the nested bends shown at right, alternating above and below the diagonal, to obtain the partition 11 10 9 8 6 2.





For $n = 14$ and $k = 3$, Sylvester's bijection is:
931^2, 751^2, 73^21, 731^4, 5^231, 53^21^3, 531^6
851, 7421, 752, 941, 6431, 842, 10 3 1

Exponential Generating Functions

Definition: A partition of a finite set S is a collection of disjoint, nonempty subsets of S, called blocks, whose union is S. Let $B(n)$ denote the number of partitions of the set $[n] = \{1, 2, \ldots, n\}$, and let $S(n, k)$ denote the number of partitions of the set $[n]$ into k blocks.

Example: The partitions of the set [3] are

$$1|2|3,\ 1|23,\ 2|13,\ 3|12,\ 123,$$

so the Bell number $B(3) = 5$, and the Stirling numbers $S(3, 3) = 1$, $S(3, 2) = 3$, and $S(3, 1) = 1$.

Theorem: The exponential generating functions for Stirling numbers are

$$\sum_{n \geq 0} S(n, k) \frac{x^n}{n!} = \frac{(e^x - 1)^k}{k!}$$

The Exponential Generating Function for Set Partitions

Corollary: The exponential generating function for Bell numbers is

$$\sum_{n\geq 0} B(n)\frac{x^n}{n!} = e^{e^x-1}.$$

Calculate the first few terms of e^{e^x-1}.

$e^{x+\frac{x^2}{2}+\frac{x^3}{6}+\cdots}$

$$= 1 + (x + \tfrac{x^2}{2} + \tfrac{x^3}{6} + \cdots) + \frac{(x+\frac{x^2}{2}+\frac{x^3}{6}+\cdots)^2}{2} + \cdots$$

$$= 1 + x + (\tfrac{x^2}{2} + \tfrac{x^2}{2}) + (\tfrac{x^3}{6} + \tfrac{x^3}{2} + \tfrac{x^3}{6}) + \cdots$$

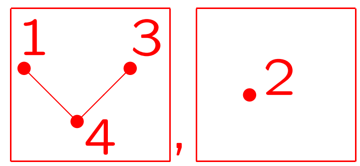
$$= 1 + x + (1+1)\tfrac{x^2}{2} + (1+3+1)\tfrac{x^3}{6} + \cdots.$$

Why does $\displaystyle\sum_{n\geq 0} S(n,k)\frac{x^n}{n!} = \frac{(x+\frac{x^2}{2}+\frac{x^3}{6}+\cdots)^k}{k!}$?

The Exponential Formula

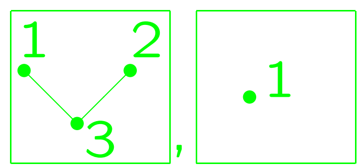
Partitions of $[n]$ and labelled n-vertex graphs are two examples of sets of hands. Each hand is a set of cards

$\boxed{14}, \boxed{2}, \boxed{3}$



that are templates labelled with $[n]$

$\boxed{12}, \boxed{1}, \boxed{1}$



chosen, with repetition, from decks D_i, $i \geq 1$. For set partitions, each deck D_i, $i \geq 1$, has one template $\boxed{12 \cdots i}$.

Theorem: The numbers $H(n,k)$, which count hands whose k cards use the labels in $[n]$, have exponential generating functions

$$\sum_{n \geq 0} H(n,k) \frac{x^n}{n!} = \frac{D(x)^k}{k!}$$

where $D(x)$ is the exponential generating function for the deck cardinalities.

Understanding the Exponential Formula

Idea of the proof: For $k = 2$, interpret the coefficient of $x^n/n!$ in the product

$$\left(\sum_{i \geq 1} d_i \frac{x^i}{i!} \right)^2 = \sum_{n \geq 2} \left(\sum_{i=1}^{n-1} \binom{n}{i} d_i d_{n-i} \right) \frac{x^n}{n!}$$

as the number of ordered $[n]$-labelled hands with 2 cards. Select the number i of labels for the first card, then independently choose a label set for the first card, a template for the first card and a template for the second card.

Examples: Set partitions with 2 blocks have exponential generating function

$$\frac{(x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots)^2}{2!} = 1\frac{x^2}{2!} + 3\frac{x^3}{3!} + 7\frac{x^4}{4!} + \cdots.$$

Labelled graphs with 2 connected components have exponential generating function

$$\frac{(x + \frac{x^2}{2!} + 4\frac{x^3}{3!} + \cdots)^2}{2!} = 1\frac{x^2}{2!} + 3\frac{x^3}{3!} + 19\frac{x^4}{4!} + \cdots.$$

The Principle of Inclusion-Exclusion

Theorem: If α and β map subsets of a finite set to elements of an abelian group, then

$$\alpha(S) = \sum_{T \subseteq S} \beta(T), \text{ for all } S$$

if and only if

$$\beta(S) = \sum_{T \subseteq S} (-1)^{|S-T|} \alpha(T), \text{ for all } S.$$

Example: To deduce the familiar formula

$$\begin{aligned} |A_1 \cup A_2 \cup A_3| = & |A_1| + |A_2| + |A_3| \\ & - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3| \\ & + |A_1 \cap A_2 \cap A_3|, \end{aligned}$$

for subsets A_i of a finite set A, define

$$\begin{aligned} \beta(T) &= |\{x \in A \mid x \notin A_i \Leftrightarrow i \in T\}| \\ \alpha(S) &= |\{x \in A \mid x \notin A_i \Rightarrow i \in S\}|, \end{aligned}$$

then calculate $\beta([3]) = |A| - |A_1 \cup A_2 \cup A_3|$ using the above formula for $\beta(S)$.

Multinomial Coefficients

Definition: For $S = \{s_1, s_2, \ldots, s_\ell\}_< \subseteq [n-1]$,

$$\binom{n}{s_1\ s_2{-}s_1 \cdots n{-}s_\ell} = \frac{n!}{s_1!(s_2-s_1)!\cdots(n-s_\ell)!}.$$

The binomial coefficient $\binom{n}{k} = \dfrac{n!}{k!(n-k)!}$ is the number of cardinality k subsets of $[n]$ and

$$\binom{n}{s_1\ s_2{-}s_1 \cdots n{-}s_\ell} = \binom{n}{s_\ell}\binom{s_\ell}{s_{\ell-1}}\cdots\binom{s_2}{s_1},$$

so this multinomial coefficient is the number of sequences of ℓ nested subsets of $[n]$ whose cardinalities are the numbers in S.

Lemma: This multinomial coefficient is the number of permuations of $[n]$ whose descent set is contained in S.

Example: If $n = 9$ and $S = \{2, 5, 7\}$, then $\{4, 6\} \subset \{1, 3, 4, 6, 7\} \subset \{1, 3, 4, 6, 7, 8, 9\}$ corresponds to $\pi = 461378925$, whose descent set is $\{2, 7\}$.

Permutations with a Given Descent Set

Theorem: For $S = \{s_1, s_2, \ldots, s_\ell\}_< \subseteq [n-1]$, the number of permutations of $[n]$ with descent set equal to S is

$$\sum_{1 \le i_1 < i_2 < \cdots < i_k \le \ell} (-1)^{\ell-k} \binom{n}{s_{i_1}\ s_{i_2}-s_{i_1}\ \cdots\ n-s_{i_k}}.$$

Proof: The multinomial coefficient

$$\alpha(S) = \binom{n}{s_1\ s_2-s_1\ \cdots\ n-s_\ell}$$

is the number of permutations of $[n]$ whose descent set is contained in S. For $T \subseteq [n-1]$, if $\beta(T)$ is the number of permutations of $[n]$ whose descent set equals T, then

$$\alpha(S) = \sum_{T \subseteq S} \beta(T).$$

for all S. By the principle of inclusion-exclusion,

$$\beta(S) = \sum_{T \subseteq S} (-1)^{|S-T|} \alpha(T).$$

References

George Andrews, The Theory of Partitions.

Richard Stanley, Enumerative Combinatorics.

Herbert Wilf, Generatingfunctionology.

J. J. Sylvester, "A constructive theory of partitions, arranged in three acts, an interact and an exodion", *Amer. J. Math* **5** (1882), 251–330.

Doron Zeilberger, "Proof of the alternating sign matrix conjecture". (**Abstract:** The number of $n \times n$ matrices whose entries are either -1, 0, or 1, whose row and column sums are all 1, and such that in every row and every column the non-zero entries alternate in sign, is proved to be $[1!4!\cdots(3n-2)!]/[n!(n+1)!\cdots(2n-1)!]$ as conjectured by Mills, Robbins, and Rumsey.)

Related Talks at the October Workshop

George Andrews, "MacMahon's partition analysis and computer algebra".

Mireille Bousquet-Melou, "Lecture hall partitions".

Ira Gessel, Counting labeled binary trees by ascents and descents.

Christian Krattenthaler, Determinant evaluations and plane partitions enumeration.

Greg Kuperberg, "Enumeration using statistical mechanics".

Dennis Stanton, "Trigonometric positivity and lattice paths".

Doron Zeilberger, "Computational enumerative linguistics".