A combinatorial application of Alexander duality

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Abstract

The Möbius number of a finite partially ordered set equals (up to sign) the difference between the number of even and odd edge covers of its incomparability graph. One way to deduce this formula uses Stanley's combinatorial Alexander duality theorem for Eulerian posets and Rota's cross-cut theorem for lattices. Thereby, the formula may be viewed as a consequence of two theorems from algebraic topology: Alexander Duality and the Nerve Theorem. We use these theorems to obtain a refinement that relates the homology of a poset's order complex to the cohomology of its incomparability complex, whose simplices are sets of edges of its incomparability graph that do not cover.

0. Introduction

Recall that if $\hat{P} = P \cup \{\hat{0}, \hat{1}\}$ is a finite poset, the vertices of its incomparability graph G are the elements of P and the edges of G are the 2-element antichains in P. The Möbius number $\mu_{\hat{P}}(\hat{0}, \hat{1})$ is, by Philip Hall's theorem, the reduced Euler characteristic $\tilde{\chi}(\Delta)$ of the order complex $\Delta = \Delta(P)$, whose simplices are the nonempty chains in P.

Proposition 0.1 [2] *For a finite poset P*

$$\mu_{\hat{P}}(\hat{0}, \hat{1}) = (-1)^{|P|-1} \sum_{k \geq 0} (-1)^k N_k$$

where N_k is the number of k-subsets of edges that cover all the vertices in G.

Proof: If $P \neq \emptyset$ is a chain then the Möbius number is 0 and $N_k = 0$ for all k. If $P = \emptyset$, then the Möbius number is -1 and the only nonzero N_k is $N_0 = 1$.

If P is not a chain, let $\hat{B}$ be the Boolean algebra of all subsets of P. Let $(\hat{B} - \Delta)^*$ be abstract simplicial complex whose simplices are the complements of sets of $\hat{B} - \Delta$. So $(\hat{B} - \Delta)^*$ is pure complex whose facets are the complements of 2-element antichains of P.

We use an observation: for any abstract simplicial complex C, from the Inclusion-Exclusion principle, it follows

$$\tilde{\chi}(C) = \sum_{k \geq -1} (-1)^k f_j(C) = \sum_{k \geq 0} (-1)^k |\{k\text{−subset of facets of } C \text{ with empty intersection}\}|$$

where $f_j(C)$ denotes the number of j-dimensional simplices in C.

Hence $\tilde{\chi}((\hat{B} - \Delta)^*) = \sum_{k \geq 0} (-1)^k |\{k\text{-subset of 2-element antichains of } P \text{ covering all the elements of } P\}|$. But $f_j(\Delta) + f_{|P|-j-2}((\hat{B} - \Delta)^*) = \binom{|P|}{j+1}$ for all $-1 \leq j \leq |P| - 1$, $\tilde{\chi}(\Delta) = (-1)^{|P|-1} \tilde{\chi}((\hat{B} - \Delta)^*)$. But $\tilde{\chi}((\hat{B} - \Delta)^*) = \sum_{k \geq 0} (-1)^k |\{k\text{-subset of facets (considered as subsets of } P) \text{ of } (\hat{B} - \Delta)^* \text{ covering all the elements of } P\}|$; facets of $(\hat{B} - \Delta)^*$ are 2-element antichains of P, so the proposition follows from Hall's theorem.

Some illustrations are given in [2].

Another proof deduces the proposition above from Stanley's Alexander duality theorem for Eulerian posets and Rota's cross-cut theorem: Suppose P is not a chain and let $|P| = m+2$. Let $\hat{B}$ be again the Boolean algebra of all subsets of P and let Q be the poset of nonempty chains in P. The atoms in the lattice $\hat{B} - Q$ are the 2-element antichains in P, so by Rota's cross-cut theorem [4]

$$\mu_{\hat{B}-Q}(\hat{0}, \hat{1}) = \sum_{k \geq 0} (-1)^k N_k$$

where N_k is the number of k-subsets of edges that cover the incomparability graph of P. By Stanley's combinatorial Alexander duality theorem [6]

$$\mu_{\hat{Q}}(\hat{0}, \hat{1}) = (-1)^{|P|-1} \mu_{\hat{B}-Q}(\hat{0}, \hat{1}). \tag{1}$$

The order complex of Q is the first barycentric subdivision of the order complex of P, so Hall's theorem implies $\mu_{\hat{P}}(\hat{0}, \hat{1}) = \mu_{\hat{Q}}(\hat{0}, \hat{1})$.

This proof led immediately to the realization that Propostion 0.1 is a consequence of Alexander duality, since (1) can be deduced from Lemma 1.2 below and the Euler-Poincaré formula [4]. Thereby, Alexander duality emerged as a topological tool that can be used to obtain combinatorial information about any finite poset, not just subposets of Eulerian, Cohen-Macaulay posets.

1. The incomparability complex

The lemmas below capture the topology underlying Stanley's combinatorial Alexander duality theorem and Rota's cross-cut theorem. The coefficient group for reduced homology and cohomology is a field. The empty complex is a (-1)-dimensional complex whose reduced homology and cohomology vanish except in dimension -1, when they are 1-dimensional vector spaces over the field of coefficients.

Lemma 1.2 [5] *Let $\hat{B} = B \cup \{\hat{0}, \hat{1}\}$ be the Boolean algebra of subsets of a set with $m+2$ elements. If $\hat{Q} = Q \cup \{\hat{0}, \hat{1}\}$ is any subposet of $\hat{B}$, then*

$$\tilde{H}_k(\Delta(Q)) \cong \tilde{H}^{m-k-1}(\Delta(B - Q))$$

for all k.

Proof: Alexander duality for the m-sphere $|\Delta(B)|$, the geometric realization of $\Delta(B)$, yields

$$\tilde{H}_k(|\Delta(Q)|) \cong \tilde{H}^{m-k-1}(|\Delta(B)| - |\Delta(Q)|).$$

Note that $|\Delta(B - Q)|$ is a strong deformation retract of $|\Delta(B)| - |\Delta(Q)|$.

The second lemma is proved using the Nerve Theorem of algebraic topology.

Lemma 1.3 [1] *Let $\hat{L} = L \cup \{\hat{0}, \hat{1}\}$ be a finite lattice. Let $\Gamma(L)$ be the cross-cut complex, whose simplices are the nonempty sets of minimal elements in L that do not have join $\hat{1}$. Then $|\Delta(L)|$ and $|\Gamma(L)|$ are homotopy equivalent.*

We use these lemmas to prove the following homological refinement of Proposition 0.1.

Definition 1.4 The *incomparability complex* Γ of a finite poset $\hat{P} = P \cup \{\hat{0}, \hat{1}\}$ is the simplicial complex of nonempty sets of 2-element antichains in P whose union is not all of P.

Theorem 1.5 *Let* $\hat{P} = P \cup \{\hat{0}, \hat{1}\}$ *with order complex* Δ *and incomparability complex* Γ*. If* P *is not a chain and* $|P| = m + 2$*, then*

$$\tilde{H}_k(\Delta) \cong \tilde{H}^{m-k-1}(\Gamma)$$

for all k*.*

Proof: Let $\hat{B}$ be the Boolean algebra of all subsets of P and let Q be the poset of nonempty chains in P. Since $\Delta(Q)$ is the first barycentric subdivision of Δ, $\tilde{H}_k(\Delta) = \tilde{H}_k(\Delta(Q))$. By Stanley's Lemma 1.2, $\tilde{H}_k(\Delta(Q)) = \tilde{H}^{m-k-1}(\Delta(B - Q))$. By Björner's Lemma 1.3, $\tilde{H}^{m-k-1}(\Delta(B-Q)) = \tilde{H}^{m-k-1}(\Gamma)$ since Γ is the cross-cut complex $\Gamma(B-Q)$.

When P is not a chain, the Euler-Poincaré formula [3] and Theorem 1.4 imply that $\tilde{\chi}(\Delta) = (-1)^{|P|-1}\tilde{\chi}(\Gamma)$. Proposition 0.1 is thereby a corollary of the above theorem.

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