Finitely axiomatizable omega-categorical theories

by

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Abstract

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Let $\mathcal{T}$ be the class of complete, finitely axiomatizable, ω-categorical theories. We prove three non-structure results for $T \in \mathcal{T}$.

Definition. Say that a 1-type p satisfies the *Mazoyer hypothesis* if there is $k \in \omega$ such that whenever $a \models p$ and $a \in \mathrm{acl}(bC)$, there are $c_1, \ldots, c_{k-1} \in \mathrm{acl}(C)$ so that $a \in \mathrm{acl}(b, c_1, \ldots, c_{k-1})$.

Theorem (A). *If $T \in \mathcal{T}$ and some 1-type satisfies the Mazoyer hypothesis, then T has the order property.*

All known examples of $T \in \mathcal{T}$ have a type satisfying the Mazoyer hypothesis.

The following theorem provides hypotheses that are sufficient for the strict order property.

Theorem (B). *Let $T \in \mathcal{T}$. Suppose that there is a 1-type p so that*

1. *algebraic closure restricted to p is a trivial pregeometry, and*

2. *p satisfies the following degeneracy condition: for all $a \models p$, if $a \in \mathrm{acl}(C)$, then there is $c \in C$ such that $a \in \mathrm{acl}(c)$.*

Then, T has the strict order property.

Theorem (B) generalizes a result of Ivanov: if $T \in \mathcal{T}$ has a distributive lattice of algebraically closed sets, then T has the strict order property.

Finally, by altering the hypotheses of Theorem (B), we get sufficient conditions for the tree property.

Theorem (C). *Let $T \in \mathcal{T}$. Suppose there is a 1-type p so that*

1. *algebraic closure restricted to p is a trivial geometry,*
2. *p satisfies the Mazoyer hypothesis,*
3. *p is stably embedded.*

Then, T has the tree-property (i.e. T is not simple).

Professor Leo A. Harrington
Dissertation Committee Chair

In memory of Sidney T. Lippel,

who taught: Slow and steady wins the race.

Contents

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Chapter 1

Introduction and Background

1.1 Introduction

Let $\mathcal{T}$ be the class of all complete, finitely axiomatizable ω-categorical theories. Two familiar theories in $\mathcal{T}$ are the theory of a dense linear order and the theory of atomless Boolean algebra. Other examples are given in Chapter 3.

The basic question addressed in this dissertation is, what can be said about the structure of $T \in \mathcal{T}$?

Conjecture 1.1.1 (Macpherson [16]). If $T \in \mathcal{T}$, then T has the strict order property.

To understand the motivation for the conjecture, note that the finitely many axioms of T must enforce that models are infinite. Since T is ω-categorical, algebraic closure is locally finite; thus, function symbols cannot be used to make the models infinite. The only other known technique for finitely axiomatizing a scheme of infinity is to use a dense ordering.

One strategy for proving Macpherson's conjecture is to first weaken the conclusion:

Conjecture 1.1.2 (Order property conjecture). If $T \in \mathcal{T}$, then T has the order property (i.e. T is unstable).

We show that the order property conjecture holds under an additional hypothesis on the algebraic closure operator of T.

Theorem 1.1.3. *If $T \in \mathcal{T}$ and there is a non-algebraic 1-type satisfying the* Mazoyer hypothesis*, then T has the order property.*

Definition 1.1.4. A 1-type p satisfies the *Mazoyer hypothesis* if there is $k \in \omega$ so that whenever $a \models p$ and $a \in \operatorname{acl}(bC)$, there are $c_1, \ldots, c_{k-1} \in \operatorname{acl}(C)$ so that $a \in \operatorname{acl}(b, c_1, \ldots, c_{k-1})$.

All known examples in $\mathcal{T}$ have a type satisfying the Mazoyer hypothesis; in fact, they all have the following stronger, global property.

Definition 1.1.5. A theory has *bounded algebraic arity* if there is $k \in \omega$ so that $a \in \operatorname{acl}(bC)$ implies that there are $c_1, \ldots, c_{k-1} \in \operatorname{acl}(C)$ so that $a \in \operatorname{acl}(b, c_1, \ldots, c_{k-1})$.

For refinements of the definitions and further discussion, see Chapter 4.

Major work in stability theory has already established weaker versions of the order property conjecture. Zilber proved that $T \in \mathcal{T}$ is not uncountably categorical [31, 32]. (See Zilber's book [33] for a complete exposition). Cherlin, Harrington, and Lachlan improved Zilber's result, showing that $T \in \mathcal{T}$ is not ω-stable [7]. Subsequently, Hrushovski constructed ω-categorical theories that are stable but not ω-stable [10]. (See [29] and [3] for published accounts of the construction). Thus, the order property conjecture reduces to the question of whether $T \in \mathcal{T}$ can be stable but not ω-stable. The only known examples of stable, non-ω-stable, ω-categorical theories are those constructed by Hrushovski, using his method of dimensions, with a standard predimension function $\delta(A) = |A| - \sum_{R \in L} \alpha_R |R^A|$. Theories constructed in this way have bounded algebraic arity (see Example 2.2.17); thus, by Theorem 1.1.3, they are not finitely axiomatizable.

The following special case of the order property conjecture sheds light on what remains to be proved.

Theorem 1.1.6 (Ivanov & Macpherson [14], Theorem 4.11). *If $T \in \mathcal{T}$ and there are* infinitely many finite simple groups involved in T*, then T has the order property.*

The notion of a finite simple group involved in a theory is due to Hrushovski [12]; see Chapter 5 for the definition and a proof of Theorem 1.1.6. The collection $\mathcal{G}(T)$

of finite simple groups involved in T is a measure of the complexity of the algebraic closure operator of T. If $\mathcal{G}(T)$ is infinite, then the algebraic closure is "complicated". So, Theorem 1.1.6 says that in the complicated case, $T \in \mathcal{T}$ has the order property. On the other hand, the existence of a Mazoyer type is a simplifying hypothesis on the algebraic closure, and Theorem 1.1.3 establishes the order property in this case. Thus, in both the simple and complicated cases, $T \in \mathcal{T}$ has the order property. It remains to analyze theories that are "in between" simple and complicated (if there are any). It is worth repeating that all known examples of $T \in \mathcal{T}$ have bounded algebraic arity. Bounded algebraic arity implies that $\mathcal{G}(T)$ is finite (Proposition 5.2.7), so in particular, no examples with infinite $\mathcal{G}(T)$ are known.

We now return to consideration of Macpherson's conjecture. Ivanov has verified the conjecture in a special case:

Theorem 1.1.7 (Ivanov [13], Theorem 3.1). *If $T \in \mathcal{T}$ and the lattice of algebraically closed subsets of T is distributive, then T has the strict order property.*

The distributive hypothesis is very restrictive; for example, atomless Boolean algebras do not have a distributive lattice of closed sets. Using the ideas from the proof of Theorem 1.1.3, we can weaken the distributivity hypothesis somewhat.

Theorem 1.1.8. *Let $T \in \mathcal{T}$. Suppose that there is a non-algebraic 1-type p so that algebraic closure restricted to p is a pregeometry and p is* 1-Mazoyer. *Then, T has the strict order property.*

The definition of *1-Mazoyer* is given in Section 2.1. See Example 2.2.8 for an argument that distributivity implies the hypotheses of Theorem 1.1.8. If $T_1 \in \mathcal{T}$ has a distributive lattice of closed sets and $T_2 \in \mathcal{T}$ does not, then the disjoint union of T_1 and T_2 is an example of a theory in $\mathcal{T}$ which does not have a distributive lattice of closed sets but satisfies the hypotheses of Theorem 1.1.8. However, it is not clear whether any *interesting* non-distributive examples in $\mathcal{T}$ satisfy the hypotheses.

By altering the argument for Theorem 1.1.8, we get hypotheses that are sufficient for the tree property:

Theorem 1.1.9. *Let $T \in \mathcal{T}$. Suppose there is a* 1*-type p so that*

1. *algebraic closure restricted to p is a trivial geometry,*

2. *p satisfies the Mazoyer hypothesis, and*

3. *p is stably embedded.*

Then, T has the tree-property (i.e. T is not simple).

The structure of the dissertation is as follows. The remainder of this chapter contains sections on notation and background material. Chapter 2 contains discussion of the Mazoyer hypothesis and algebraic arity. Chapter 3 gives a catalogue of examples of theories in $\mathcal{T}$; as far as possible, the algebraic structure of each example is described. Chapter 4 contains proofs of the non-structure results mentioned above, i.e. Theorem 1.1.3, Theorem 1.1.8, and Theorem 1.1.9. Chapter 5 is about the notion of a finite simple group involved in a theory and includes a proof of Theorem 1.1.6. Finally, Chapter 6 lists some questions for further investigation.

1.2 Notational conventions

For the most part, our notation is standard. Perhaps the greatest potential sources of confusion are Convention 1.2.1 and Convention 1.2.2.

The symbol T will always denote a complete theory with infinite models. To each theory T, we associate a highly saturated model $\mathfrak{M}$, which we take as the universal domain for T. The symbols A,B,C, etc. denote subsets of $\mathfrak{M}$ with cardinality less than $|\mathfrak{M}|$; a,b,c, etc. denote elements of $\mathfrak{M}$; and $\vec{a}$, $\vec{b}$, etc. denote finite tuples from $\mathfrak{M}$. The symbols X,Y and Z etc. represent subsets of $\mathfrak{M}$ with any cardinality.

Convention 1.2.1. We work in $\mathfrak{M}$, not $\mathfrak{M}^{eq}$. Thus, acl() is the algebraic closure in $\mathfrak{M}$, and we make a distinction between elements of $\mathfrak{M}$ and tuples from $\mathfrak{M}$. We explicitly mention those situations where we need to use $\mathfrak{M}^{eq}$.

Convention 1.2.2. If T is ω-categorical, we take $\mathfrak{M}$ to be the countable model. Hence, in this case A,B,C are *finite* sets.

If T is finitely axiomatizable, let $\mathrm{var}(T)$ denote the number of variables appearing in some fixed finite axiomatization of T.

Convention 1.2.3. If T is finitely axiomatizable, we call a finite subset *small* if it has cardinality less than $\mathrm{var}(T)$. A *small type* is a type with with small domain.

An *m-formula* is a formula with at most m free variables. Specifying further, an *(m, n)-formula* is a $(m+n)$-formula whose free variables are partitioned into exactly m object variables and at most n parameter variables. For any $n > 0$, let Δ_n denote the complete set of $(1, n)$-formulas. (Notice: we do not close Δ_n under conjunctions.) We will be interested in Δ_n in the case that T is ω-categorical; in this case, of course, Δ_n is finite.

If T is ω-categorical, then T is uniformly locally finite. Let $\beta : \omega \to \omega$ be the function that witness this property, i.e. $\beta(n) = \max\{|\operatorname{acl}(a_1, \ldots, a_n)| : a_1, \ldots, a_n \in \mathfrak{M}\}$.

1.3 Background

1.3.1 Combinatorial properties of formulas

A set of formulas is called *k-inconsistent* if every subset of cardinality k is inconsistent.

Fix a theory T; recall that we work inside of a universal domain $\mathfrak{M}$. The following combinatorial properties of formulas are all due to Shelah; see [26].

Definition 1.3.1. Let $\phi(\vec{x}, \vec{y})$ be a (m, n)-formula. In the definitions below, the tuples $\vec{a}_i$ have length m and the tuples $\vec{b}_i$ have length n.

1. ϕ has the *order property* if there are infinite sequence $\langle \vec{a}_i : i \in \omega \rangle$ and $\langle \vec{b}_i : i \in \omega \rangle$ so that $\mathfrak{M} \models \phi(\vec{a}_i, \vec{b}_j)$ if and only if $i \geq j$. (A formula is *stable* if it does not have the order property.)

2. ϕ has the *strict order property* if there is an infinite sequence $\langle \vec{b}_i : i \in \omega \rangle$ so that the sets $\phi(\mathfrak{M}^m, \vec{b}_i)$ form an infinite descending chain:

$$\phi(\mathfrak{M}^m, \vec{b}_0) \supsetneq \phi(\mathfrak{M}^m, \vec{b}_1) \supsetneq \cdots \supsetneq \phi(\mathfrak{M}^m, \vec{b}_i) \supsetneq \cdots$$

3. ϕ has the *independence property* if there is $\langle \vec{b}_i : i \in \omega \rangle$ such that for each $S \subseteq \omega$, there is $\vec{a}_S$ so that $\mathfrak{M} \models \phi(\vec{a}_S, \vec{b}_i)$ if and only if $i \in S$.

4. ϕ has the *tree property with respect to* k if there are parameters $\{\vec{b}_\sigma : \sigma \in \omega^{<\omega}\}$ such that

   (a) for each branch $f \in \omega^\omega$, the set of formulas $\{\phi(x, \vec{b}_\sigma) : \sigma \in \omega^{<\omega}, \sigma \subset f\}$ is consistent, and

   (b) for each $\sigma \in \omega^{<\omega}$, the set of formulas $\{\phi(x, \vec{b}_{\sigma \frown i}) : i \in \omega\}$ is k-inconsistent.

   We say that ϕ has the *tree property* if there is k so that ϕ has the tree property with respect to k.

For each of the four properties defined above, we say that a theory T has the property if some formula of T has the property. Also, T is *stable* if no formula of T has the order property, and T is *simple* if no formula of T has the tree property.

We now state some relationships between these properties. The first of these relationships is immediate from the definitions.

Proposition 1.3.2. *If a formula has the strict order property or the independence property then it has the order property.*

The next proposition is a deeper combinatorial result.

Proposition 1.3.3 (Shelah [26], Theorem II.4.7). *A formula $\phi(\vec{x}, \vec{y})$ has the order property if and only if either ϕ has the independence property or some conjunction $\bigwedge_i \phi(\vec{x}, \vec{y}_i) \wedge \bigwedge_j \neg\phi(\vec{x}, \vec{y}_j)$ has the strict order property.*

The strict order property corresponds to the usual notion of order, as the following proposition shows.

Proposition 1.3.4. *The theory T has the strict order property if and only if T interprets a partial order with an infinite chain.*

Proof. The right to left implication is immediate: the formula with the strict order property is the formula $x < y$, where $<$ is the interpreted partial order. For the converse, suppose that the (m, n)-formula $\phi(\vec{x}, \vec{y})$ has the strict order property. Define the equivalence relation $\sim$ so that $\vec{y}_1 \sim \vec{y}_2$ if the formulas $\phi(\vec{x}, \vec{y}_1)$ and $\phi(\vec{x}, \vec{y}_2)$ are equivalent. Then, the quotient $\mathfrak{M}^n/\sim$ is partially ordered by

$$[\vec{y}_1/\sim] < [\vec{y}_2/\sim] \Longleftrightarrow \phi(\mathfrak{M}, \vec{y}_1) \subsetneqq \phi(\mathfrak{M}, \vec{y}_2).$$

Since ϕ has the strict order property, the partial order $<$ has an infinite chain. □

Proposition 1.3.5. *If a formula has the strict order property, then it has the tree property with respect to 2.*

Proof. The formula with the tree property is $y_1 < x < y_2$, which defines an interval in the interpreted partial order. Using compactness, there are infinitely many pairwise disjoint intervals; these form the first level of the tree. By compactness again, divide each of these intervals into infinitely many pairwise disjoint sub-intervals to form the second level of the tree. Continuing in this way, one builds the full ω-branching tree. □

The following proposition is due to Shelah [27]; the proof can also be found in Kim [15, Proposition 2.20].

Proposition 1.3.6. *If T has the tree property, then T has the order property.*

To summarize, the following implications hold among the properties: the strict order property implies the tree property, which implies the order property, which is equivalent to either the independence property or the strict order property. The first two implications are not reversible.

1.3.2 Dividing and rank

The following definitions are also due to Shelah [27]. See [15] or [30] for more information on dividing and simple theories.

Definition 1.3.7. A formula $\phi(\vec{x}, \vec{a})$ *divides over* B *with respect to* k if there is an infinite sequence $\langle \vec{a}_i : i \in \omega \rangle$ in $tp(\vec{a}/B)$ such that $\{\phi(x, \vec{a}_i) : i \in \omega\}$ is k-inconsistent. A formula *divides over* B if it divides over B with respect to some k. We sometimes abbreviate ϕ *divides over* B *with respect to* k by ϕ k*-divides over* B.

Definition 1.3.8. Let Δ be a finite set of formulas and let $k \geq 2$. Let $\Gamma(\vec{x})$ be a partial type over B. We define the *dividing rank* $D_k^\Delta(\Gamma(\vec{x}))$ as follows.

1. $D_k^\Delta(\Gamma(\vec{x})) \geq 0$ if $\Gamma(x)$ is consistent.

2. $D_k^\Delta(\Gamma(\vec{x})) \geq n + 1$ if there are $\delta(\vec{x}, \vec{y}) \in \Delta$ and a tuple $\vec{a}$ so that $D_k^\Delta(\Gamma(x) \cup \{\delta(\vec{x}, \vec{a})\}) \geq n$ and $\delta(\vec{x}, \vec{a})$ divides over B with respect to k.

In the standard way, we set $D_k^\Delta(\Gamma(x)) = n$ if $D_k^\Delta(\Gamma(x)) \geq n$, but $D_k^\Delta(\Gamma(x)) \not\geq n+1$. We write $D_k^\Delta(\Gamma(x)) = \infty$ if $D_k^\Delta(\Gamma(x)) \geq n$ for all n.

It is easy to check that the following proposition holds.

Proposition 1.3.9. *If* Δ *includes the formula* $x = y$*, then for all* $k \geq 2$*,* $D_k^\Delta(\Gamma(x)) = 0$ *if and only if* $\Gamma(x)$ *is algebraic.*

If $\Delta = \{\phi\}$, we write $D_k^\phi()$ for $D_k^\Delta()$. The following proposition is immediate from the definitions.

Proposition 1.3.10. *The formula* ϕ *has the tree property with respect to* k *if and only if* $D_k^\phi(x = x) = \infty$.

Thus, a theory is simple if and only if all ranks D_k^ϕ are finite. Moreover, we can code a finite set of formulas into a single formula, so T is simple if and only if all ranks D_k^Δ are finite.

1.3.3 Stable embedding

We use the notion of *stable embedding* in Section 4.5.

Definition 1.3.11. A set $S \subseteq \mathfrak{M}$ is *stably embedded (in $\mathfrak{M}$)* if for each set of the form $S^n \cap X$, where X is parameter-definable subset of $\mathfrak{M}^n$, there is an $X' \subset \mathfrak{M}^n$ that is definable *over parameters in* S such that $S^n \cap X = S^n \cap X'$. A type p is *stably embedded* if the set $p(\mathfrak{M})$ is stably embedded.

The following result is proved using the definability of types in stable theories.

Proposition 1.3.12. *If T is stable, then every subset of $\mathfrak{M}$ is stably embedded.*

It is certainly possible to have stably embedded sets when the theory is not stable. For example, the theory of an existentially closed difference field is simple and non-stable, and the fixed field is stably embedded [6].

1.3.4 Axiomatizations

There are two natural generalizations of finite axiomatizability that are relevant to this dissertation. The first is *finite-variable axiomatizability.* We say that a theory T is *finite-variable axiomatizable* if there is a set of axioms S for T so that only finitely many variable symbols appear in S. Of course, a finitely axiomatizable theory is finite-variable axiomatizable. Hedman has studied finite-variable axiomatizations and variants [8]. It turns out that, in the ω-categorical case, finite-variable axiomatizability is not really a generalization of finite axiomatizability after all.

Theorem 1.3.13 (Hedman [8], Theorem 1.22). *An ω-categorical theory is finitely axiomatizable if and only if it is finite-variable axiomatizable.*

The second generalization is *finite axiomatizability modulo a universal theory.* Let $\mathcal{T}'$ be the class of ω-categorical theories axiomatized by a set of sentences of the form $U \cup \{\sigma\}$, where the sentences in U are universal. All of the results in this dissertation still hold if $\mathcal{T}$ is replaced with $\mathcal{T}'$ everywhere. However, I do not know any examples that are in $\mathcal{T}'$ but not in $\mathcal{T}$.

Chapter 2

Arity of algebraic closure

In this chapter, we define a notion of arity for the algebraic closure operator of a theory, providing a general framework for the Mazoyer hypothesis. We discuss examples relating to algebraic arity in Section 2.2.

2.1 Definitions

A closure operator is a very familiar object; however, we repeat the definition here in order to fix names for the defining properties.

Definition 2.1.1. A *closure operator* on a set S is a map $\mathrm{cl}() : \mathcal{P}(S) \to \mathcal{P}(S)$ that is

1. *Monotone:* $X \subseteq Y$ implies $\mathrm{cl}(X) \subseteq \mathrm{cl}(Y)$,
2. *Idempotent:* $\mathrm{cl}(\mathrm{cl}(X)) = \mathrm{cl}(X)$, and
3. *Increasing:* $X \subseteq \mathrm{cl}(X)$.

The closure operator that interests us is the algebraic closure of a theory T. We also consider more general operators of the form $\mathrm{acl}() \cap V$, where $V \subseteq \mathfrak{M}$. Note that if $V \neq \mathfrak{M}$, then $\mathrm{acl}() \cap V$ is not a closure operator; it is idempotent and monotone, but not increasing.

We recall the following common terminology.

Definition 2.1.2. A closure operator cl() is *disintegrated* if whenever $x \in \mathrm{cl}(Y)$, there is $y \in Y$ such that $x \in \mathrm{cl}(y)$. The definition makes sense for an operator of the form $\mathrm{cl}() \cap V$, too.

We are now ready for the main definition.

Definition 2.1.3. We define the *arity* of the operator $\mathrm{acl}() \cap V$ as follows.

1. We say that $\mathrm{acl}() \cap V$ is *1-ary* if it is disintegrated.
2. For $k > 1$, say that $\mathrm{acl}() \cap V$ is *k-ary* if whenever $x \in \mathrm{acl}(yZ) \cap V$, there are $z_1, \dots, z_{k-1} \in \mathrm{acl}(Z)$ such that $x \in \mathrm{acl}(y, z_1, \dots, z_{k-1})$.

We say that $\mathrm{acl}() \cap V$ has *bounded arity* if it is k-ary for some k.

The following is a "global" definition of arity for a theory.

Definition 2.1.4. The theory T has *algebraic arity* k if acl() is k-ary. As in the introduction, T has *bounded algebraic arity* if acl() is k-ary for some k.

And we make the following definition of arity for a type.

Definition 2.1.5. A 1-type p is *k-Mazoyer* if $\mathrm{acl}() \cap p(\mathfrak{M})$ is k-ary. And we say that p is *Mazoyer* if $\mathrm{acl}() \cap p(\mathfrak{M})$ has bounded arity.

It is important to note the difference between the definition of arity and the following notion.

Definition 2.1.6 (Vassiliev [28]). Say that the operator $\mathrm{acl}() \cap V$ is *k-degenerate* if whenever $x \in \mathrm{acl}(Y) \cap V$, there are $y_1, \dots, y_k \in Y$ so that $x \in \mathrm{acl}(y_1, \dots, y_n)$.

Clearly, 1-degeneracy is the same as having arity 1, and k-degeneracy implies arity k, but the converse is not true. For example, let T be the theory of a vector space over a finite field. Algebraic closure is linear span, and it has arity 2: if $x \in \mathrm{acl}(y, z_1, \dots, z_n)$, then there are scalars $\alpha_1, \dots \alpha_n$ such that $x \in \mathrm{acl}(y, \sum_i \alpha_i z_i)$. However, acl() is not k-degenerate for any k: let $Y = \{y_0, \dots, y_k\}$ be a linearly independent set; then, $\sum_i y_i$ is algebraic in Y, but not in any proper subset.

Algebraically closed fields provide an example of a theory without bounded algebraic arity; see Example 2.2.12.

We now introduce another operator derived from acl() which further clarifies the relation between k-degeneracy and k-arity.

Definition 2.1.7. Define $\mathrm{acl}_k()$ by the formula

$$\mathrm{acl}_k(X) = \bigcup_{Y \subseteq X, |Y| \leq k} \mathrm{acl}(Y).$$

Write $\mathrm{acl}_k^n()$ for the result of composing $\mathrm{acl}_k()$ with itself n times.

In general, $\mathrm{acl}_k()$ is not idempotent; however, $\mathrm{acl}_k^\omega()$ is idempotent and is a closure operator. Note that $\mathrm{acl}_k^\omega() \subseteq \mathrm{acl}()$.

The next proposition follows immediately from the definitions.

Proposition 2.1.8. *For all $V \subseteq \mathfrak{M}$, the operator $\mathrm{acl}() \cap V$ is k-degenerate if and only if $\mathrm{acl}_k() \cap V = \mathrm{acl}() \cap V$.*

Proposition 2.1.9. *If algebraic closure is k-ary, then $\mathrm{acl}_k^\omega() = \mathrm{acl}()$.*

(Notice that the proposition is about the full algebraic closure, not $\mathrm{acl}() \cap V$.)

Proof. Suppose that acl() has arity k. Take $x \in \mathrm{acl}(Y)$. We need to show that $x \in \mathrm{acl}_k^\omega(Y)$. Proceed by induction on $|Y|$. If $|Y| \leq k$, there is noting to prove. Otherwise, write Y as yZ. Using the k-ary assumption, choose $z_1, \ldots, z_{k-1} \in \mathrm{acl}(Z)$ so that $x \in \mathrm{acl}(y, z_1, \ldots, z_{k-1})$. By induction hypothesis, each $z_i \in \mathrm{acl}_k^\omega(Z) \subseteq \mathrm{acl}_k^\omega(yZ)$. Thus, $y, z_1, \ldots, z_{k-1} \in \mathrm{acl}_k^\omega(yZ)$, and we see that $x \in \mathrm{acl}_k^\omega(yZ)$. □

We use Proposition 2.1.9 in the proof of Proposition 5.2.7 below, which shows that if T has bounded algebraic arity, then there are finitely many finite simple groups involved in T.

2.2 Examples

As mentioned in the introduction, every known example of $T \in \mathcal{T}$ has bounded algebraic arity; see Chapter 3 for examples in $\mathcal{T}$. In this section we will work without the hypothesis of finite axiomatizability. The section is divided into two parts. In the first part, we give some abstract properties that imply bounded arity or the Mazoyer hypothesis. In the second part, we discuss some specific kinds of theories with bounded algebraic arity.

2.2.1 Stronger properties

Example 2.2.1 (Modular lattice of closed sets). Recall: a lattice is modular if $X \leq Z$ implies $(X \vee Y) \wedge Z = X \vee (Y \wedge Z)$.

Proposition 2.2.2. *If an ω-categorical theory has a modular lattice of algebraically closed sets, then it has bounded algebraic arity. Specifically,* acl() *has arity* $(\beta(2)-1)$.

Proof. In the lattice of closed sets, $X \wedge Y = X \cap Y$, and $X \vee Y = \operatorname{acl}(X \cup Y)$.

Let B be a closed set and take $a \in \operatorname{acl}(cB) \setminus B$. Using modularity, we have

$$a \in (\operatorname{acl}(c) \vee B) \cap \operatorname{acl}(ac) = \operatorname{acl}(c) \vee (B \cap \operatorname{acl}(ac)) = \operatorname{acl}(c \cup (B \cap \operatorname{acl}(ac))).$$

Since $a \notin B$ and $c \notin B$, $|B \cap \operatorname{acl}(ac)| < \beta(2) - 1$, and we are done. □

Example 2.2.3 (Modular pregeometry). First, let us recall the definitions of (combinatorial) geometries and pregeometries.

Definition 2.2.4. A closure operator cl() is a *pregeometry* if it satisfies the *exchange axiom:* if $a \in \operatorname{cl}(bC) \setminus \operatorname{cl}(C)$, then $b \in \operatorname{cl}(aC)$. A pregeometry is a *geometry* if $\operatorname{cl}(\emptyset) = \emptyset$ and $\operatorname{cl}(a) = \{a\}$ for all a.

A *modular* pregeometry is a pregeometry with a modular lattice of closed sets. (This characterization is equivalent to the more common definition in terms of the "modular law" for dimension.)

Suppose that acl() is a modular pregeometry. From the above example, we know that if T is ω-categorical, then acl() has arity $(\beta(2) - 1)$. In fact, without the hypothesis of ω-categoricity, we get the following stronger result.

Proposition 2.2.5. *If* acl() *is a modular pregeometry, then it is* 2*-ary.*

Proof. Take $B = \mathrm{acl}(B)$ and $a \in \mathrm{acl}(cB) \setminus B$. We may assume that $a \notin \mathrm{acl}(c)$. As in Proposition 2.2.2, we know that $a \in \mathrm{acl}(c \cup (B \cap \mathrm{acl}(ac)))$. Note that $(B \cap \mathrm{acl}(ac)) \not\subseteq \mathrm{acl}(c)$, since otherwise we would have $a \in \mathrm{acl}(c)$. Choose $d \in (B \cap \mathrm{acl}(ac)) \setminus \mathrm{acl}(c)$. Applying the exchange axiom, $a \in \mathrm{acl}(cd)$, as desired. □

Below, we define two well-known properties of pregeometries.

Definition 2.2.6.

1. A pregeometry cl() is *locally modular* if there is an element a so that $\mathrm{cl}_a()$ is modular, where $\mathrm{cl}_a()$ is the pregeometry defined by the formula $\mathrm{cl}_a(X) = \mathrm{cl}(Xa)$. The process of forming $\mathrm{cl}_a()$ from cl() is called *localizing at a*.

2. A pregeometry cl() on a set S is *homogeneous* if for all closed sets B and distinct elements $a_1, a_2 \notin B$, there is an automorphism of $(S, \mathrm{cl}())$ mapping a_1 to a_2. (An automorphism of $(S, \mathrm{cl}())$ is a bijection on S that respects cl().)

Proposition 2.2.7. *If algebraic closure is a locally modular and homogeneous pregeometry, then it is* 3*-ary.*

Proof. Take $a \in \mathrm{acl}(cB)$. We may assume that $B \not\subseteq \mathrm{acl}(\emptyset)$. Localize at any element $b_0 \in B \setminus \mathrm{acl}(\emptyset)$. Since the pregeometry is homogeneous, localizing at b_0 has the same effect as localizing at any other non-algebraic element. Thus, $\mathrm{acl}_{b_0}()$ is a modular pregeometry. Now apply Proposition 2.2.5. □

Example 2.2.8 (Distributive lattice of closed sets). Suppose that the lattice of algebraically closed sets is a distributive lattice. A distributive lattice is modular, so Proposition 2.2.2 applies. However, we can get a better bound on the arity, as shown in the following proposition.

Proposition 2.2.9. *Let T be an ω-categorical theory, and suppose that the lattice of algebraically closed sets is distributive. Then, the arity of* acl() *is* $\beta(1)$.

Proof. Let B be a closed set and take $a \in \mathrm{acl}(cB) \setminus B$. Then, distributing $\wedge$ over $\vee$, we have

$$a \in (\mathrm{acl}(c) \vee B) \cap \mathrm{acl}(a) = (\mathrm{acl}(c) \cap \mathrm{acl}(a)) \vee (B \cap \mathrm{acl}(a)) \subseteq \mathrm{acl}(c \cup (B \cap \mathrm{acl}(a))).$$

Since $a \notin B$, $|B \cap \mathrm{acl}(a)| < \beta(1)$. □

Following Ivanov [13], we can reduce the arity to one by considering $\mathrm{acl}() \cap p(\mathfrak{M})$ for a particular type p.

Proposition 2.2.10. *Let T be ω-categorical. If the lattice of algebraically closed sets is distributive, then there is a type $p \in S_1(\emptyset)$ that is 1-Mazoyer.*

Proof. By ω-categoricity, there is an element $a_0 \notin \mathrm{acl}(\emptyset)$ so that $\mathrm{acl}(a_0)$ is minimal, i.e. if $b \in \mathrm{acl}(a_0) \setminus \mathrm{acl}(\emptyset)$, then $\mathrm{acl}(b) = \mathrm{acl}(a_0)$. Let $p = tp(a_0)$. Suppose that $a \in \mathrm{acl}(B) \cap p(\mathfrak{M})$, where $B = \{b_1, \ldots, b_n\}$. Then, by distributivity, we have

$$a \in \mathrm{acl}(B) \cap \mathrm{acl}(a) = \mathrm{acl}((\mathrm{acl}(b_1) \cap \mathrm{acl}(a)) \cup \cdots \cup (\mathrm{acl}(b_n) \cap \mathrm{acl}(a))).$$

Since $a \notin \mathrm{acl}(\emptyset)$, there is an index i so that $\mathrm{acl}(b_i) \cap \mathrm{acl}(a) \nsubseteq \mathrm{acl}(\emptyset)$. Choosing $d \in (\mathrm{acl}(b_i) \cap \mathrm{acl}(a)) \setminus \mathrm{acl}(\emptyset)$, we have that $a \in \mathrm{acl}(d)$, by the minimality of $\mathrm{acl}(a)$. Since $d \in \mathrm{acl}(b_i)$, we conclude that $a \in \mathrm{acl}(b_i)$. □

Remark 2.2.11. The type p in the proof above has the additional property that acl() restricted to p is a pregeometry. Take a,b and C in $p(\mathfrak{M})$, and suppose that $a \in \mathrm{acl}(bC) \setminus \mathrm{acl}(C)$. By the proposition, $a \in \mathrm{acl}(b)$, and in fact, $a \in \mathrm{acl}(b) \setminus \mathrm{acl}(\emptyset)$. By the minimality of $\mathrm{acl}(b)$, we see that $b \in \mathrm{acl}(a)$. Thus, the exchange axiom holds.

2.2.2 Theories

Example 2.2.12 (Strongly minimal theories). Let T be a strongly minimal theory. Then, the algebraic closure of T is a homogeneous pregeometry. We say that T is *locally modular* if its algebraic closure is locally modular. It is common terminology to call T *pseudomodular* if T has bounded algebraic arity. (Also, one says that T is *pseudoprojective* if T is pseudomodular, but not disintegrated.)

Theorem 2.2.13 (Buechler [4], Theorem A). *Any pseudomodular strongly minimal theory is locally modular.*

In Example 2.2.3, we showed that a locally modular, homogeneous pregeometry is 3-ary. Thus, the theorem says that if the algebraic closure in a strongly minimal set is k-ary for some k, then it is actually 3-ary.

Since an algebraically closed field is strongly minimal but not locally modular, it gives an example of a theory which does not have bounded algebraic arity.

Example 2.2.14 (ω-stable, ω-categorical theories). The Mazoyer hypothesis for a type p was originally formulated in the ω-stable, ω-categorical context, in the case that p is strongly minimal [17]. Poizat gave the hypothesis its name (*l'hypothese d'Mazoyer*) in [20, §F]. Mazoyer used the hypothesis to prove the following special case of the non-finite axiomatizability of ω-stable, ω-categorical theories.

Theorem 2.2.15 (Mazoyer [17]). *If T is ω-stable, ω-categorical and there is a strongly minimal type p that satisfies the Mazoyer hypothesis, then T is not finitely axiomatizable.*

A proof of Mazoyer's theorem, due to Lascar, is sketched in [20]. (It also appears in [22]).

Saffe [22] showed that any ω-stable, ω-categorical theory has a strongly minimal type that satisfies a modified form of the Mazoyer hypothesis.

Theorem 2.2.16 (Saffe [22]). *Let T be ω-stable, ω-categorical. Let p be a strongly minimal type. There are $m, k \in \omega$ such that if $a \in \operatorname{acl}(bC) \cap p(\mathfrak{M})$ and C is m-saturated, then there are $c_1, \ldots, c_{k-1} \in \operatorname{acl}(C)$ such that $a \in \operatorname{acl}(b, c_1, \ldots, c_{k-1})$.*

Saffe's variant of the Mazoyer hypothesis is still sufficient for Mazoyer's proof. However, Saffe's proof of Theorem 2.2.16 uses the full strength of the results from [7] on the fine structure of ω-stable, ω-categorical theories, so we do not get a "quick" proof of non-finite axiomatizability.

Example 2.2.17 (Generic Structures). Fix a finite relational language L. For the purposes of this example, we call an L-structure *generic* if it is constructed by Hrushovski's method of dimensions, using a "standard" predimension function. We show below that a generic structure has bounded algebraic arity; in fact, algebraic closure is k-degenerate, for some k. But first, we sketch the relevant aspects of the construction; for full details, see [29] or [3].

Associated to the generic structure is a real-valued function δ on finite sets, which is called the predimension function and is of the form $\delta(A) = |A| - \sum_{R\in L} \alpha_R |R^A|$, where each $\alpha_R \in \mathbb{R}$. The structure is constructed in such a way that

$$a \in \operatorname{acl}(B) \text{ if and only if } \delta(aB) < \delta(B). \tag{2.1}$$

The complete theory of a generic structure is stable.[1]

By (2.1), we have that

$$a \in \operatorname{acl}(B) \iff \delta(aB) - \delta(B) < 0 \iff 1 - \sum_{R\in L} \alpha_R |R^{aB} \setminus R^B| < 0.$$

Thus, a is algebraic in B because there are "too many" relations between a and B.

Proposition 2.2.18. *Let $k = \sum_{R\in L} \lfloor (1 + \frac{1}{\alpha_R}) \rfloor (arity(R) - 1)$. Suppose that $a \in \operatorname{acl}(B)$. Then, there is $B_0 \subset B$ so that $|B_0| \leq k$ and $\sum_{R\in L} \alpha_R |R^{aB_0} \setminus R^{B_0}| > 1$. Hence, $a \in \operatorname{acl}(B_0)$, and $\operatorname{acl}()$ is k-degenerate.*

Proof. Note that $\lfloor (1 + x) \rfloor$ is the least integer strictly greater than x.

First suppose that there is an $R \in L$ so that $\alpha_R |R^{aB} \setminus R^B| > 1$. Then, $|R^{aB} \setminus R^B| \geq \lfloor (1 + \frac{1}{\alpha_R}) \rfloor$. In this case, we can choose a minimal $B_0 \subseteq B$ so that $|R^{aB_0} \setminus R^{B_0}| \geq \lfloor (1 + \frac{1}{\alpha_R}) \rfloor$. Note that $|B_0| \leq \lfloor (1 + \frac{1}{\alpha_R}) \rfloor (arity(R) - 1)$. By the choice of B_0, we have $\alpha_R |R^{aB_0} \setminus R^{B_0}| > 1$.

On the other hand, suppose that for each R, we have $\alpha_R |R^{aB} \setminus R^B| \leq 1$. Then, $|R^{aB} \setminus R^B| < \lfloor (1 + \frac{1}{\alpha_R}) \rfloor$, so there are at most $\lfloor (1 + \frac{1}{\alpha_R}) \rfloor (arity(R) - 1)$ elements from B involved in R-relations with a. Call this set of elements B_R. Then, $\alpha_R |R^{aB_R} \setminus R^{B_R}| = \alpha_R |R^{aB} \setminus R^B|$. Letting $B_0 = \bigcup_R B_R$, we have $|B_0| \leq k$, and

$$\sum_{R\in L} \alpha_R |R^{aB_0} \setminus R^{B_0}| = \sum_{R\in L} \alpha_R |R^{aB} \setminus R^B| > 1.$$

□

Since the generic construction produces a stable theory, by Theorem 1.1.3, the theory is not finitely axiomatizable.

[1] There is a variant of the construction where $<$ is replaced with $\leq$ in (2.1). Proposition 2.2.18 applies equally well in this case; note, however, that the variant need not produce a stable theory.

Chapter 3

Examples of finitely axiomatizable ω-categorical theories

In this chapter, we give some examples of finitely axiomatizable ω-categorical theories. None of the examples are new. As far as possible, we describe the algebraic structure of each example.

3.1 Nuclear theories

Rosenstein classified the ω-categorical linear orders and proved that they are all finitely axiomatizable [21]. Schmerl made the following definition in order to generalize Rosenstein's work.

Definition 3.1.1 (Schmerl [23]). A theory T is *n-nuclear* if for all $p \in S_1(B)$, where B is finite, there is $B_0 \subseteq B$ such that $|B_0| \leq n$ and $p \upharpoonright B_0$ implies p. We say that T is *nuclear* of it is n-nuclear for some n.

The basic examples of nuclear theories are linear orders: any linear order is 2-nuclear (see, for example, Hodges [9, Lemma A.6.8]). On the other hand, we have the following partial converse.

Proposition 3.1.2. *If T is ω-categorical and nuclear, then T has the strict order property.*

Proof. Suppose that T is n-nuclear. Suppose we have a chain

$$q_0(\mathfrak{M}) \supsetneqq q_1(\mathfrak{M}) \supsetneqq \cdots \supsetneqq q_l(\mathfrak{M}),$$

where each $q_i \in S_1(C_i)$ is non-algebraic and $|C_i| \leq n$. Choose $a_l \models q_l$, and let $p \in S_1(a_l C_l)$ be a non-algebraic extension of q_l. Then, we can find $C_{l+1} \subseteq a_l C_l$ so that $|C_{l+1}| \leq n$ and $p \restriction C_{l+1}$ implies p. Set $q_{l+1} = p \restriction C_{l+1}$. Note that $q_l(\mathfrak{M}) \supsetneqq q_{l+1}(\mathfrak{M})$.

By ω-categoricity and compactness, there is an infinite descending chain of q_i's. By ω-categoricity again, there are only finitely many types $q(x, \vec{y}) \in S_{1+n}(\emptyset)$, so some such q appears infinitely many times in the chain. The formula isolating this q has the strict order property. □

The important point about nuclearity is that it gives a sufficient condition for finite axiomatizability:

Theorem 3.1.3 (Schmerl [23]). *Suppose that T is ω-categorical and T has a finite language. If T is nuclear, then T is finitely axiomatizable.*

The proof uses the nuclearity of T to finitely axiomatize the Scott sentence for T.

3.1.1 Examples of nuclear theories

For easy examples of nuclear theories, we can generalize the linear orders. One way to generalize is to add unary predicates. Fix a language with a binary relation symbol $<$ and unary predicates P_i, $i \in \lambda$. A λ-*order* is a structure in this language in which $<$ is interpreted as a linear order. For an ω-categorical example, take a dense linear order and name a sequence of dense, co-dense sets. The same argument that shows that each linear order is 2-nuclear also shows that each λ-order is 2-nuclear.

Another way to generalize linear orders is consider local orders [5]. One can show that a local order is 4-nuclear. The dense local order is a well-known finitely axiomatizable ω-categorical theory.

There are other classes of nuclear partial orders. For example, in [23] Schmerl shows that finitely branching trees are nuclear. Thus, ω-categorical, finitely branching trees are finitely axiomatizable, and, in fact, these are the only finitely axiomatizable

ω-categorical trees. In this context, a *tree* is a partially ordered set P such that, for each $x \in P$, the set $\{y \in P : y < x\}$ is linearly ordered. Note that such a tree need not be rooted, nor well-founded. Thus, we must take some care in defining what *finitely branching* means for P. First, we can think of P as a directed graph where there is an arrow between x and y if $x < y$. If X is subset of P, then the *components* of X are simply the connected components of X in this graph. Given $x, y \in P$, define T_{xy} to be the set $\{z : \forall w((w \leq x \wedge w \leq y) \rightarrow w < z)\}$. We say that P is *n-branching* if each T_{xy} has at most n components; P is *finitely branching* if it is n-branching for some n.

3.1.2 Properties of nuclear theories

The following observation about the algebraic closure of a nuclear theory follows immediately from the definition of nuclearity.

Proposition 3.1.4. *If T is n-nuclear, then T has n-degenerate algebraic closure.*

In the sections below, we show that certain theories are not nuclear. To do so, we use the following proposition.

Proposition 3.1.5. *If T is n-nuclear, then there is no indiscernible set of size $(n+2)$ in $\mathfrak{M}$.*

Proof. Suppose that $\{a_1, \ldots, a_{n+2}\}$ is an indiscernible set. Consider the type $p(x) = tp(a_{n+2}/a_1, \ldots, a_{n+1})$. Since T is n-nuclear, we know that p is implied by the restriction of p to some n-subset of $\{a_1, \ldots, a_{n+1}\}$. Since $\{a_1, \ldots, a_{n+2}\}$ is indiscernible, we actually have that $q(x) = tp(a_{n+2}/a_1, \ldots, a_n)$ implies $p(x)$. Using indiscernibility again, we see that $a_{n+1} \models q(x)$. Thus, $a_{n+1} \models p(x)$. However, this is impossible, because $p(x)$ contains the formula $x \neq a_{n+1}$. □

We end our discussion of nuclear theories by mentioning a result about groups interpretable in nuclear theories. Poizat [19] proves that no infinite group is interpretable in a λ-order. The proof of Poizat's result that is found in Hodges [9, Theorem A.6.9, p. 710] is essentially a proof of the following stronger statement.

Proposition 3.1.6. *If T is nuclear, then T does not interpret an infinite group.*

3.2 Partial orders of finite width

The *width* of a partial order is the maximum cardinality of an antichain. Let P be an ω-categorical partial order of finite width. Schmerl shows that the complete theory of P is finitely axiomatizable [24]. We can easily prove a special case here.

Proposition 3.2.1. *Let P be an ω-categorical partial order of width $w \in \omega$. Let T be the complete theory of P, and suppose that T admits elimination of quantifiers. Then T is $4w$-nuclear, and hence, T is finitely axiomatizable.*

Proof. Take a finite subset B of P and a type $p \in S_1(B)$. Let $a \models p(x)$. We may assume that $a \notin B$. Decompose B into three pieces: $B_< = \{b \in B : b < a\}$, $B_> = \{b \in B : b > a\}$, and $B_\| = \{b \in B : b \| a\}$, where $b \| a$ means that b and a are incomparable.

Let $A_<$ be the antichain of maximal elements of $B_<$, $A_>$ be the antichain of minimal elements of $B_>$, and $A_\|^{\min}$, $A_\|^{\max}$ be the antichains of minimal and maximal elements of $B_\|$. Then, let $B_0 = A_< \cup A_> \cup A_\|^{\min} \cup A_\|^{\max}$. Note that $|B_0| \leq 4w$. It is clear that $p \restriction B_0$ implies $p \restriction (B_< \cup B_>)$. Moreover, $p \restriction B_0$ implies $p \restriction B_\|$, since if an element x is comparable to some element in $B_\|$, then x is comparable to some element in $A_\|^{\min} \cup A_\|^{\max}$. Therefore, $p \restriction B_0$ implies p, and we have shown that T is $4w$-nuclear. □

Remark 3.2.2. The proof of nuclearity given above makes essential use of quantifier elimination and the fact that the language is binary, since we are using the property that any type $p \in S_1(B)$ is implied by $\bigcup_{b \in B} p \restriction \{b\}$.

For each w, there is a partial order P_w satisfying the hypotheses of Proposition 3.2.1: the class $\mathcal{C}_w$ of finite partial orders of width w has the amalgamation property, so P_w is the Fraïssé limit of $\mathcal{C}_w$. Presumably, there are ω-categorical partial orders of finite width that do not admit elimination of quantifiers; Schmerl's full result tells us that they are finitely axiomatizable. Any non-nuclear example could potentially have an interesting algebraic closure operator.

Schmerl also proves that a partial order of finite width (not necessarily ω-categorical) does not have the independence property [25].

3.3 Fraïssé limits

See Hodges [9, §§7.1 & 7.4] for the details of the Fraïssé limit construction. A countable structure $\mathfrak{M}$ is the Fraïssé limit of its class of finite substructures if and only if the theory of $\mathfrak{M}$ is ω-categorical and admits elimination of quantifiers.

Any Fraïssé limit $\mathfrak{M}$ can be axiomatized by a "standard" set of axioms $\Gamma_\forall \cup \Gamma_{\forall\exists}$, where $\Gamma_\forall$ is a set of universal axioms that describe the substructures of $\mathfrak{M}$, and $\Gamma_{\forall\exists}$ is a set of universal-existential axioms that express the embedding property of $\mathfrak{M}$. If the theory T of $\mathfrak{M}$ is finitely axiomatizable, then some finite subset of $\Gamma_\forall \cup \Gamma_{\forall\exists}$ axiomatizes T. It is easy to come up with examples where $\Gamma_\forall$ is finitely axiomatizable but $\Gamma_{\forall\exists}$ is not: for example, the random graph. On the other hand, as remarked in Section 1.3.4, I have no counterexample to refute the conjecture that if $\Gamma_{\forall\exists}$ is finitely axiomatizable, then T itself is finitely axiomatizable. A finite axiomatization of $\Gamma_{\forall\exists}$ is quite a rare phenomenon.

Example 3.3.1 (Partial order). Let $\mathfrak{M}$ be the Fraïssé limit of the class of finite partial orders, and let T be the complete theory of $\mathfrak{M}$. Albert and Burris [1] prove that T is finitely axiomatizable. This theory is not nuclear, since any antichain is an indiscernible set.

We sketch the argument that T is finitely axiomatizable. First note that there is an obvious finite axiomatization of $\Gamma_\forall$; we need only finitely axiomatize $\Gamma_{\forall\exists}$. To do so, we must say that every finite partial order occurring in $\mathfrak{M}$ can be extended by one more point to any possible larger configuration. That is, we need to say that every 1-type over a finite set is realized.

First, we consider a particular family of 1-types as an illustration. For each n, let $p_n(x, \vec{y}) \in S_n(\emptyset)$ be the type of an n-element antichain in $\mathfrak{M}$. Fix $n > 1$, and let $\vec{b}$ realize p_{n-1}. Consider $p(x) = p_n(x, \vec{b}) \in S_1(\vec{b})$. Notice that p is not determined by its restriction to any proper subset of $\vec{b}$. However, there are auxiliary points c and d and a type $q(x)$ over $\{c, d\}$ so that q implies p. Namely, let c be an upper bound for $\vec{b}$ and d be a lower bound; then, p is implied by $x \not\leq c \wedge x \not\geq d$. To ensure that p is realized, it is is enough to ensure that q is realized. We take the axiom σ that says that each 1-type over a domain of size two is realized. Then, for any $n > 1$ and any choice of $\vec{b}$ realizing p_{n-1}, the axiom σ guarantees the existence of points c and d for $\vec{b}$, since we can find upper and lower bounds of pairs. Furthermore, σ guarantees that the auxiliary type q is realized. Thus, σ finitely axiomatizes the fact that each finite antichain can be extended by one element. There is no need for an infinite scheme of axioms, with one axiom to handle each possible cardinality of $\vec{b}$.

For the general finite axiomatizability argument, we consider an arbitrary type p over an arbitrary (finite) set B. We need to say that p is realized. As in Proposition 3.2.1, we decompose B into three sets $B_<$, $B_>$ and $B_\|$ according to p. At most four auxiliary points are needed: a lower bound c for $B_>$, an upper bound d for $B_<$, and lower and upper bounds e and f for $B_\|$. There is a type q over $\{c, d, e, f\}$ which implies p. We can ensure that q is realized with the axiom that says that all 1-types over a domain of size four are realized. Thus, it remains to show that we can finitely axiomatize the existence of the four auxiliary points (over any B). Note that there is some subtlety here: the auxiliary points must inter-relate in an appropriate way. For example, c must be greater than d but not less than f. See [1] for a list of axioms that guarantee the existence of the auxiliary points.

It is easy to see that T has bounded algebraic arity. Indeed, because the class of finite partial orders has free amalgamation, $\mathfrak{M}$ has *no algebraicity*, i.e. for any set X, $\mathrm{acl}(X) = X$.

Example 3.3.2 (Semilattice). Let $\mathfrak{M}$ be the Fraïssé limit of the class of all finite meet-semilattices, and let T be the theory of $\mathfrak{M}$. Albert and Burris [1] show that T is finitely axiomatizable in much the same way that they show that the theory in Example 3.3.1 is finitely axiomatizable.

The theory T is not nuclear, since $\mathfrak{M}$ has an infinite indiscernible subset. Namely, choose $b \in M$ and let $A = \{a_i\}_{i\in\omega}$ be a set of elements such that $a_i \wedge a_j = b$ for all $i \neq j$. Then, A is indiscernible.

Proposition 3.3.3. *The algebraic closure of* $\mathfrak{M}$ *is* 2*-ary.*

Proof. In $\mathfrak{M}$, the algebraic closure of a set C is the substructure generated by C. Thus, if $B = \mathrm{acl}(B)$ and $a \in \mathrm{acl}(cB) \setminus B$, then we have $a = c \wedge \bigwedge_i b_i$, for some b_i's in B. But then, $a \in \mathrm{acl}(c, (\bigwedge_i b_i))$, where $(\bigwedge_i b_i) \in B$. □

Proposition 3.3.4. *The lattice of algebraically closed sets of $\mathfrak{M}$ is not modular.*

Proof. Take elements a and b in $\mathfrak{M}$ so that $a \wedge b \neq a, b$. Let $X = \mathrm{acl}(a)$, $Y = \mathrm{acl}(b)$ and $Z = \mathrm{acl}(a, a \wedge b)$. Then, in the lattice of closed sets, we have $X < Z$ and

$$(X \vee Y) \wedge Z = \mathrm{acl}(a, b) \cap Z = \{a, b, a \wedge b\} \cap \{a, a \wedge b\} = Z.$$

However, we also have

$$X \vee (Y \wedge Z) = X \vee (\{b\} \cap \{a, a \wedge b\}) = X \vee \emptyset = X.$$

Thus, $(X \vee Y) \wedge X \neq X \vee (Y \wedge Z)$, and the lattice of closed sets is not modular. □

Example 3.3.5 (Boolean algebra). The ω-categorical completions of the theory of Boolean algebra are the theories T_n, $n \in \omega$, where T_n says that there are exactly n atoms. (Thus, T_0 is the theory of atomless Boolean algebra.) It is well known that each theory T_n is finitely axiomatizable. Let $\mathfrak{M}_n$ be the countable model of T_n.

None of the theories T_n is nuclear. One way to see this is to note that there is an infinite indiscernible set in each $\mathfrak{M}_n$. Another way: an infinite Boolean group is interpretable in $\mathfrak{M}_n$.

If $n > 0$, then $\mathfrak{M}_n$ is isomorphic to the Cartesian product of $\mathfrak{M}_0$ and n copies of the two-element Boolean algebra. Thus, each $\mathfrak{M}_n$ is interpretable in $\mathfrak{M}_0$.

Proposition 3.3.6. *Each T_n has 3-ary algebraic closure.*

Proof. Note first that the algebraic closure of a finite set B is the Boolean subalgebra generated by B and the n atoms of $\mathfrak{M}_n$. Take B so that $B = \mathrm{acl}(B)$, and suppose $a \in \mathrm{acl}(Bc)$. Since the atoms of $\mathfrak{M}_n$ are contained in B, we have that $\mathrm{acl}(Bc)$ is simply the substructure generated by Bc. Each atom of $\mathrm{acl}(Bc)$ is either of the form $b \wedge c$ or of the form $b \wedge \bar{c}$, where b is an atom of B. Since a is a join of atoms of $\mathrm{acl}(Bc)$, there are subsets $\{b_i\}_i$ and $\{b_j\}_j$ of B so that $a = \bigvee_i (b_i \wedge c) \vee \bigvee_j (b_j \wedge \bar{c})$. Distributing $\wedge$ over $\vee$, we see that $a = ((\bigvee_i b_i) \wedge c) \vee ((\bigvee_j b_j) \wedge \bar{c})$. Thus, $a \in \mathrm{acl}(c, (\bigvee_i b_i), (\bigvee_j b_j))$, where $(\bigvee_i b_i), (\bigvee_j b_j) \in B$. □

As is noted in [13, Prop. 1.4], the lattice of algebraically closed sets of T_n is not distributive.

Chapter 4

Results

In this chapter, we prove the results mentioned in the introduction. The theorems give sufficient conditions for the order property, the strict order property, and the tree property; moreover, they give some information about the number of parameter variables appearing in the formula that witnesses each property.

The structure of the chapter is as follows. Section 4.1 contains a coordinatization result for $T \in \mathcal{T}$. The coordinatization involves the introduction of a set of extra parameters. In order to make use of coordinatization, we must minimize the effect of these parameters; this is the role of the Mazoyer hypothesis. In Section 4.2 we introduce a notion of saturation and show that saturated sets exist in the absence of the order property or the strict order property. Section 4.3 contains the proof of Theorem 1.1.3, which yields the order property; Section 4.4 contains the proof of Theorem 1.1.8, which yields the strict order property; and Section 4.5 contains the proof of Theorem 1.1.9, which yields the tree property.

Convention. Throughout this chapter, T is ω-categorical. Thus, A, B and C are finite sets.

4.1 Coordinatization

In this section, we prove a coordinatization result for $T \in \mathcal{T}$.

In the terminology of [7, §4], if $p \in S_1(\emptyset)$ is non-algebraic, and X is an infinite $\emptyset$-definable set, then p *coordinatizes* X if for all $d \in X$, there is an element $a \models p$ with $a \in \mathrm{acl}(d)$.

For our purposes, we need a variant definition that allows parameters.

Definition 4.1.1. Say that p *coordinatizes* X *over* E if

1. $p \in S_1(B)$ is a non-algebraic type,

2. X is a infinite definable set over C,

3. $B \cup C \subseteq E$,

4. $E = \mathrm{acl}(E)$,

5. $p(\mathfrak{M}) \cap E = X \cap E = \emptyset$, and

6. for all $d \in X$ there is $a \models p$ such that $a \in \mathrm{acl}(dE)$.

We say that p *coordinatizes* X when there is an E such that p coordinatizes X over E, but the specific E is unimportant. Given a 1-type r, we say that p *coordinatizes* r if p coordinatizes the definable set $r(\mathfrak{M})$.

Note that beyond the parameters used to define p and X, we allow "extra" parameters in E when taking the algebraic closure in condition 6. We will not insist that E be finite.

Remark 4.1.2. If p coordinatizes r over E and r' is a non-algebraic extension of r with $\mathrm{dom}(r') \subseteq E$, then p coordinatizes r' over E.

In [7], the "Coordinatization Theorem" shows that for an ω-stable, ω-categorical theory, one may assume that there is a rank-one $p \in S_1(\emptyset)$ which coordinatizes the whole model (in the parameter-free sense). In contrast, we show for $T \in \mathcal{T}$ that every non-algebraic $p \in S_1(B)$ coordinatizes *some* non-algebraic small type r. (Recall that a *small set* is a set with cardinality less that $\mathrm{var}(T)$, and a *small type* is a type with small domain.)

To prove the coordinatization result, we first show that we may assume $T \in \mathcal{T}$ to be model complete. We use the following technique, which appears in both [16] and [12]. Assume that the sentence σ axiomatizes T. Expand the language by adding a new relation symbol R_ϕ for each proper subformula ϕ of σ. For each atomic ϕ, add the axiom that expresses the equivalence of R_ϕ and ϕ. For each non-atomic ϕ, add the axiom which says that R_ϕ is equivalent to the formula that results from replacing each immediate subformula ψ of ϕ with R_ψ. For example, in the case that ϕ is of the form $\psi_1 \wedge \psi_2$, add the axiom that says that R_ϕ is equivalent to $R_{\psi_1} \wedge R_{\psi_2}$; and in the case that ϕ is of the form $\exists x\, \psi(x, \vec{y})$, add the axiom that says that R_ϕ is equivalent to $\exists x R_\psi(x, \vec{y})$. Let T' be the set of new axioms added in this way. Note that $T' \cup \{\sigma\}$ is finite, and $\mathrm{var}(T' \cup \{\sigma\}) = \mathrm{var}(T)$. The axioms of T' are all universal-existential. Moreover, the original axiom σ is equivalent modulo T' to a sentence with at most one quantifier. Thus, $T' \cup \{\sigma\}$ has a universal-existential axiomatization. By Lindström's test, $T' \cup \{\sigma\}$ is model complete. Therefore, replacing T with $T' \cup \{\sigma\}$, we may assume that T is model complete.

Convention 4.1.3. From now on, we will assume that $T \in \mathcal{T}$ is model complete.

The importance of model completeness is that it provides the following criterion for elementary substructures of $\mathfrak{M}$.

Proposition 4.1.4. *Let $T \in \mathcal{T}$. Let X be a subset of $\mathfrak{M}$. Then, the following are equivalent.*

1. *X is a model of T.*

2. *X is an elementary substructure of $\mathfrak{M}$.*

3. *X realizes every small 1-type over a subset of X.*

Proof. (1) implies (2) by model completeness, and (2) implies (3) because small types are isolated. For (3) implies (1): if X realizes every 1-type over a small subset, then an inductive argument shows that X and $\mathfrak{M}$ agree on sentences with at most $\mathrm{var}(T)$ many variables; hence, $X \models T$. □

The following definition allows us to give a quick proof of the coordinatization lemma.

Definition 4.1.5. Let p be a 1-type and B a set. A *p-envelope of B* is a maximal set $E \supseteq B$ satisfying the condition

$$\mathrm{acl}(E) \cap p(\mathfrak{M}) = \mathrm{acl}(B) \cap p(\mathfrak{M}).$$

Notice that if E is a p-envelope of B, then E is algebraically closed.

Lemma 4.1.6. *Let $T \in \mathcal{T}$. For each non-algebraic type $p \in S_1(B)$, there is a small non-algebraic 1-type r such that p coordinatizes r.*

Proof. Let E be a p-envelope of B. Since $\mathrm{acl}(B) \cap p(\mathfrak{M}) = \emptyset$, we know that p is not realized in E. But p is isolated; thus, E is not an elementary submodel of $\mathfrak{M}$. By Proposition 4.1.4, there is a 1-type r over a small subset of E so that r is not realized in E. Since E is an p-envelope, if $d \models r$, then there is $a \models p$ so that $a \in \mathrm{acl}(dE)$. Hence, p coordinatizes r (over E). □

Note that E in the proof above need not be finite. In the remainder of the section, we reprove Lemma 4.1.6, showing that we can get a finite E. (The ideas of the proof are related to the proof of Theorem 5.2.2 on group involvement.)

As in [12], Proposition 4.1.4 provides a way to construct an elementary substructure of $\mathfrak{M}$: build the substructure as the union of a countable chain, realizing a small 1-type at each step of the chain. A variant of this construction is to build a chain in which we realize a small, non-algebraic 1-type at each odd stage and take algebraic closures at each even stage. If the chain starts with B, and we are given $p \in S_1(B)$, then p must be realized at some stage, since p is isolated. Thus, in the course of the construction, adding a realization of a small type r "causes" p to be realized.

Lemma 4.1.7. *Let $T \in \mathcal{T}$. For each non-algebraic 1-type p, there is a small non-algebraic 1-type r and a finite set E such that p coordinatizes r over E.*

Proof. Fix a non-algebraic $p \in S_1(B)$. Build a model N over B by alternately realizing small 1-types and taking algebraic closure, omitting p as long as possible. That is, construct a chain $B = B_0 \subset B_1 \subset \cdots \subset B_n \subset \cdots$ of finite sets with the following properties.

1. If n is even, then $B_{n+1} = \mathrm{acl}(B_n)$.

2. If n is odd, then $B_{n+1} = B_n d_n$, where $d_n \models r_n(x) \in S_1(C_n)$, $C_n \subseteq B_n$, $|C_n| < \mathrm{var}(T)$, and $r_n(x)$ is not realized in B_n. (Note: r_n is necessarily non-algebraic, since it is not realized in B_n, which is algebraically closed.)

3. For each n, the element d_n is chosen from among the realizations of r_n so that $\mathrm{acl}(B_n d_n) \cap p(\mathfrak{M}) = \emptyset$, unless such a choice is not possible.

4. $\bigcup_n B_n = N$ realizes all 1-types over small subsets, and thus, N is an elementary substructure of $\mathfrak{M}$.

Since p is isolated, it cannot be omitted from N. Thus, p is realized at some stage in the construction. Let $n+1$ be the first stage at which p is realized. (Note that $n > 0$, since p is not realized in $B_1 = \mathrm{acl}(B_0)$.)

Suppose that n is odd. Then, $d_n \models p(x)$, and by condition 3 above, for each $d \models r_n(x)$, there is $a \models p(x)$ such that $a \in \mathrm{acl}(B_n d)$. Thus, p coordinatizes r_n over B_n.

Suppose that n is even. This means that p is first realized by taking the algebraic closure of $B_n = B_{n-1}d_{n-1}$. By condition 3, for all $d \models r_{n-1}(x)$, there is $a \models p(x)$ such that $a \in \mathrm{acl}(B_{n-1}d)$. Hence, p coordinatizes r_{n-1} over B_{n-1}.

□

4.2 Strong m-saturation with respect to a type

In this section we define a notion of saturation and show that finite saturated sets exist in the absence of the order property and definable saturated sets exist in the absence of the strict order property. (We do not assume that T is finitely axiomatizable anywhere in this section.)

Recall that in stability theory, a model $N \preccurlyeq \mathfrak{M}$ is *strongly κ-saturated* if for all $A \subset \mathfrak{M}$ with $|A| < \kappa$, N realizes every type in $S_{<\kappa}(A)$ that is finitely satisfied in N. (See [2, Definition III.2.32, p. 69].) Note that, in general, A is not a subset of N.

We introduce a variant of strong κ-saturation here: κ is changed to a natural number, all types are now 1-types, and the condition of finite satisfiability is replaced by the requirement that the type to be realized has infinitely many realizations in common with a given target type.

Definition 4.2.1. Let $p(x) \in S_1(B)$ be a non-algebraic type, and let $m \in \omega$. Say that X *is strongly m-saturated with respect to p* if X realizes every type $q(x) \in S_1(C)$ that satisfies the conditions

- $|C| < m$, and
- $q(x) \cup p(x)$ is non-algebraic.

Suppose that the set A is strongly m-saturated with respect to $p \in S_1(B)$. Take a non-algebraic extension $p' \in S_1(AB)$ of p. Then, AB is strongly m-saturated with respect to p'. Thus, we often deal with the case where $\mathrm{dom}(p)$ is strongly m-saturated with respect to p.

Recall that Δ_m is the set of all $(1, m-1)$-formulas of T.

Lemma 4.2.2. *If no formula in Δ_m has the order property, then for each non-algebraic $p_0 \in S_1(B_0)$, there is a non-algebraic $p(x) \in S_1(B)$ extending p_0 such that B is strongly m-saturated with respect to p. (Note: B is finite!)*

Proof. We prove the contrapositive. That is, suppose that there is a non-algebraic $p_0 \in S_1(B_0)$ with the following property: for each non-algebraic $p \in S_1(B)$ extending p_0, there is a type $q_p(x) \in S_1(C)$ such that

- $|C| < m$,
- $p(x) \cup q_p(x)$ is non-algebraic, and
- $q_p(\mathfrak{M}) \cap B = \emptyset$.

We show that there is a formula in Δ_m with the order property.

Suppose that we have sequences $a_0, \ldots, a_l$; $\vec{b}_0, \ldots, \vec{b}_l$; $q_0(x, \vec{y}), \ldots, q_l(x, \vec{y})$ such that

1. each $q_j \in S_m(\emptyset)$,
2. $\bigcup_j q_j(x, \vec{b}_j) \cup p_0(x)$ is non-algebraic, and
3. $\mathfrak{M} \models q_j(a_i, \vec{b}_j)$ if and only if $i \geq j$.

(Note: in item 3, the subscript j for q_j and $\vec{b}_j$ is the same.)

We extend the sequences in the following way. Let $B = B_0 \cup \{a_0, \ldots, a_l, \vec{b}_0, \ldots, \vec{b}_l\}$. Choose a non-algebraic $p(x) \in S(B)$ extending $\bigcup_j q_j(x, \vec{b}_j) \cup p_0(x)$. Take $q_p(x) \in S_1(C)$ as given by our supposition above. Let $\vec{b}_{l+1}$ be some ordering of C and let $q_{l+1} \in S_m(\emptyset)$ be the type such that $q_p(x) = q_{l+1}(x, \vec{b}_{l+1})$. Choose any $a_{l+1} \models p(x) \cup q_p(x)$.

Then, the conditions (1)-(3) above hold for the extended sequences. Conditions (1) and (2) are immediate. For condition (3): if $i < l+1$, then $a_i \in B$, and $\mathfrak{M} \not\models q_{l+1}(a_i, \vec{b}_{l+1})$, since $q_p(\mathfrak{M}) \cap B = \emptyset$; and if $i = l+1$, then $\mathfrak{M} \models q_j(a_{l+1}, \vec{b}_j)$ for all $j \leq l+1$, because $a_{l+1} \models p(x)$ and $p(x)$ implies $\bigcup_j q_j(x, \vec{b}_j)$.

By compactness, there are infinite sequences of a's, $\vec{b}$'s and q's. By ω-categoricity, $S_m(\emptyset)$ is finite, so some q_{j_0} appears infinitely many times in the list. Condition (3) shows that the formula isolating q_{j_0} has the order property. □

Corollary 4.2.3. *If T is stable, then for all m there is a non-algebraic $p \in S_1(B)$ such that B is strongly m-saturated with respect to p.*

We now turn to the strict order property.

Lemma 4.2.4. *Fix a non-algebraic type $p \in S_1(B)$, where $|B| \leq n$. Suppose that no formula in Δ_{n+m} has the strict order property. Then, there is a definable set $X \subsetneq p(\mathfrak{M})$ such that X is strongly m-saturated with respect to p and $p(\mathfrak{M}) \setminus X$ is infinite.*

Proof. The first case to consider is the case that for all $q \in S_1(C)$, where $|C| < m$, either $q(\mathfrak{M}) \cap p(\mathfrak{M})$ or $p(\mathfrak{M}) \setminus q(\mathfrak{M})$ is finite. In this case, by ω-categoricity, there is a bound on the size of finite sets of the form $p(\mathfrak{M}) \setminus q(\mathfrak{M})$; choose X to be a larger finite subset of $p(\mathfrak{M})$.

Suppose now that there is $q_0 \in S_1(C)$, where $|C| < m$, so that both $q_0(\mathfrak{M}) \cap p(\mathfrak{M})$ and $p(\mathfrak{M}) \setminus q_0(\mathfrak{M})$ are infinite. Pick $c_0 \models q_0$ and let $X_0 = \{c_0\} \cup (p(\mathfrak{M}) \setminus q_0(\mathfrak{M}))$. If X_0 is not strongly m-saturated with respect to p, there is $q_1 \in S_1(C_1)$ so that $|C_1| < m$, $q_1(\mathfrak{M}) \cap p(\mathfrak{M})$ is infinite, and $q_1(\mathfrak{M}) \cap X_0 = \emptyset$. Note that $q_1(\mathfrak{M}) \cap p(\mathfrak{M}) \subsetneqq q_0(\mathfrak{M}) \cap p(\mathfrak{M})$. Pick $c_1 \in q_1(\mathfrak{M}) \cap p(\mathfrak{M})$ and let $X_1 = \{c_1\} \cup (p(\mathfrak{M}) \setminus q_1(\mathfrak{M}))$. If X_1 is not strongly m-saturated with respect to p, repeat the process. The process stops with some X_i having the desired property, since otherwise, we would have a infinite nested sequence of sets of the form $q_i(\mathfrak{M}) \cap p(\mathfrak{M})$, which would contradict the absence of the strict order property in Δ_{n+m}. □

Suppose that a C-definable set X is strongly saturated with respect to a type $p \in S_1(B)$. In the following sections, we make use of X by taking a p-envelope E of XBC. If X is finite, then $E \cap p(\mathfrak{M}) = \mathrm{acl}(XBC) \cap p(\mathfrak{M})$ is finite; in particular, $p(\mathfrak{M})$ is not a subset of E. The fact that $p(\mathfrak{M})$ is not a subset of E is useful, because it means that E is not an elementary submodel of $\mathfrak{M}$: whereas $\mathfrak{M}$ thinks there is a realization of p which is not in $\mathrm{acl}(XBC)$, E does not. In the case that X is an infinite set (as generally happens in Lemma 4.2.4), we cannot immediately guarantee that $p(\mathfrak{M})$ is not a subset of $\mathrm{acl}(XBC)$. In Sections 4.4 and 4.5 we impose conditions on p in order to ensure that $p(\mathfrak{M}) \not\subseteq \mathrm{acl}(XBC)$.

4.3 The order property

Below, we prove Theorem 1.1.3 of the introduction: if $T \in \mathcal{T}$ has a Mazoyer type, then T has the order property.

Let $l_k = \beta(k) + 1$, so that l_k is a strict upper bound on the size of the algebraic closure of k elements.

Lemma 4.3.1. *Let T be ω-categorical. Suppose that the non-algebraic type $p \in S_1(B)$ is k-Mazoyer. For any $n, j \in \omega$, if B is strongly $(n + jk)$-saturated with respect to p, and p coordinatizes r, where $|dom(r)| < n$, then $D^{\Delta_k}_{l_k}(r) > j$.*

Proof. For brevity, let $\Delta = \Delta_k$, $l = l_k$.

Do induction on j. If $j = 0$, there is nothing to prove (since r is non-algebraic by the definition of coordinatization).

For the induction step, fix n, and suppose that

1. $B \subset \mathfrak{M}$ is strongly $(n + (j + 1)k)$-saturated with respect to $p(x) \in S_1(B)$, and

2. p coordinatizes r over E, where $\mathrm{dom}(r) = C$ and $|C| < n$.

Take $d \models r$. Choose $a \models p$ so that $a \in \mathrm{acl}(dE)$. Since p is k-Mazoyer, we can find $\vec{e} \in E^{k-1}$ so that $a \in \mathrm{acl}(d\vec{e})$. (If $k = 1$, this step is still valid, since $d \notin E$.)

Let $\delta(x, \vec{e}a)$ isolate $tp(d/\vec{e}a)$. Notice that $\delta(x, \vec{e}a)$ l-divides over $C\vec{e}$. (Take an infinite sequence $\langle a_i : i \in \omega\rangle$ in $tp(a/C\vec{e})$. If d' satisfies $\bigwedge_{i\in\{m_1,\ldots,m_l\}} \delta(x, \vec{e}a_i)$, then the l elements $a_{m_1}, \ldots, a_{m_l}$ are all algebraic in $d'\vec{e}$, contradicting the choice of l.)

By strong $(n+k)$-saturation of B with respect to p, we can choose $\tilde{a}\tilde{d} \models tp(ad/C\vec{e})$ so that $\tilde{a} \in B$. Let $r' = tp(\tilde{d}/C\vec{e}\,\tilde{a})$; note that $\delta(x, \vec{e}\,\tilde{a}) \in r'$. We may assume $D^{\Delta}_{l}(\tilde{d}/C\vec{e}) < \infty$; otherwise, $D^{\Delta}_{l}(r) \geq D^{\Delta}_{l}(\tilde{d}/C\vec{e}) = \infty$, and we are done. With the assumption of finite rank, we have

$$D^{\Delta}_{l}(r') < D^{\Delta}_{l}(\tilde{d}/C\vec{e}) \leq D^{\Delta}_{l}(r),$$

where the strict inequality is witnessed by the l-dividing of $\delta(x, \vec{e}\,\tilde{a})$ over $C\vec{e}$.

Note that r' is not algebraic, because no realization of r is algebraic in E. By Remark 4.1.2, we know that p coordinatizes r' over E. Also, $|\,\mathrm{dom}(r')| < n + k$. By induction, the fact that B is strongly $((n+k)+jk)$-saturated with respect to p implies that $D^{\Delta}_{l}(r') > j$. The rank inequality above establishes that $D^{\Delta}_{l}(r) > j + 1$. □

Remark 4.3.2. The above argument can be thought of as an *almost analysis* of the type r from $p(\mathfrak{M}) \cup \mathrm{dom}(p)$. The saturation of $\mathrm{dom}(p)$ guarantees that the analysis continues through enough steps to force the rank of r to be large.

Theorem 4.3.3. *Suppose that $T \in \mathcal{T}$ and a 1-type p_0 satisfies the Mazoyer hypothesis. Further suppose that $j_0 = D^{\Delta_k}_{l_k}(x = x)$ is finite. Then, some formula in $\Delta_{\mathrm{var}(T)+j_0k}$ has the order property.*

Proof. Suppose not. Then, by Lemma 4.2.2, there is $p \in S_1(B)$ extending p_0 so that B is strongly $(\mathrm{var}(T) + j_0 k)$-saturated with respect to p. Note that p is also k-Mazoyer, since the property is preserved in extensions. Since T is finitely axiomatizable, by Lemma 4.1.6, we can choose r so that p coordinatizes r and $|\operatorname{dom}(r)| < \mathrm{var}(T)$. Then, by Lemma 4.3.1, $D^{\Delta_k}_{l_k}(r) > j_0$. This contradicts the choice of j_0. □

We now get Theorem 1.1.3 as a corollary:

Corollary 4.3.4. *If $T \in \mathcal{T}$ has a 1-type that satisfies the Mazoyer hypothesis, then T has the order property.*

Proof. If not, then T is stable, hence simple, and all D^{Δ}_{l} ranks are finite. This contradicts Theorem 4.3.3. □

4.4 The strict order property

In this section we prove Theorem 1.1.8, which gives a sufficient condition for $T \in \mathcal{T}$ to have the strict order property. Theorem 1.1.8 implies Ivanov's theorem (Theorem 1.1.7); however, as mentioned in the introduction, it is not clear that Theorem 1.1.8 is really more general than Ivanov's theorem.

Definition 4.4.1. Let $p \in S_1(B)$. By *algebraic closure restricted to p* we mean the closure operator on $p(\mathfrak{M})$ defined by the formula $\mathrm{cl}(X) = \mathrm{acl}(XB) \cap p(\mathfrak{M})$, for all $X \subseteq p(\mathfrak{M})$.

Theorem 4.4.2. *Let $T \in \mathcal{T}$. Suppose that there is $p \in S_1(B)$ so that*

1. *algebraic closure restricted to p is a geometry, and*

2. *p is 1-Mazoyer.*

Then, some formula in $\Delta_{(|B|+\mathrm{var}(T)+\beta(1))}$ has the strict order property.

Proof. Note that acl() restricted to p is a *trivial* geometry, because p is 1-Mazoyer.

Suppose no formula in $\Delta_{(|B|+\mathrm{var}(T)+\beta(1))}$ has the strict order property. By Lemma 4.2.4, there is a C-definable set X so that $X \subsetneq p(\mathfrak{M})$, X is strongly $(\mathrm{var}(T)+\beta(1))$-saturated with respect to p, and $p(\mathfrak{M}) \setminus X$ is infinite.

Consider $\mathrm{acl}(XBC) \cap p(\mathfrak{M})$. If $a \in \mathrm{acl}(XBC) \cap p(\mathfrak{M})$, then, by the 1-Mazoyer assumption, either $a \in \mathrm{acl}(XB) \cap p(\mathfrak{M})$ or $a \in \mathrm{acl}(C)$. In the first case, $a \in X$, since acl() restricted to p is a trivial geometry. Thus, $\mathrm{acl}(XBC) \cap p(\mathfrak{M}) \subseteq X \cup \mathrm{acl}(C)$. Since $\mathrm{acl}(C)$ is finite, we see that $p(\mathfrak{M}) \nsubseteq \mathrm{acl}(XBC)$.

Let E be a p-envelope of XBC. There is a realization of p in $\mathfrak{M}$ that is not in $\mathrm{acl}(XBC)$, but E has no such point. Since $\mathrm{acl}(XBC)$ is definable over $BC \subseteq E$, we see that E is not an elementary submodel of $\mathfrak{M}$. Hence, by Proposition 4.1.4, there is a small type r over a subset of E so that $r(\mathfrak{M}) \cap E = \emptyset$.

Pick $d \models r$. Then, because E is a p-envelope, there is $a \models p$ so that $a \in \mathrm{acl}(dE) \setminus \mathrm{acl}(E)$. Since p is 1-Mazoyer, we know that $a \in \mathrm{acl}(d)$. By strong $\mathrm{var}(T)$-saturation of X with respect to p, we can choose $a_0 d_0 \models tp(ad/\operatorname{dom}(r))$ so that $a_0 \in X$. Note that $a_0 \in \mathrm{acl}(d_0)$.

Let $r' = tp(d_0/\operatorname{dom}(r)a_0)$, and repeat the argument in the above paragraph with r' in place of r, using the strong $(\mathrm{var}(T)+1)$-saturation of X with respect to p. The result is a pair $a_1 d_1$ so that $d_1 \models r'$, $a_0 \neq a_1$, and $a_0, a_1 \in \mathrm{acl}(d_1) \cap E$. Saturation allows this analysis to continue until there are distinct elements $a_0, \ldots, a_{\beta(1)} \in \mathrm{acl}(d_{\beta(1)})$. This contradicts the definition of $\beta(1)$. □

In the following corollary (which is Theorem 1.1.8 from the introduction), we weaken the hypothesis that algebraic closure restricted to p is a geometry. Note, however, that in doing so we lose information about the number of variables in the formula with the strict order property.

Corollary 4.4.3. *Let $T \in \mathcal{T}$. Suppose that there is $p \in S_1(B)$ so that*

1. *algebraic closure restricted to p is a pregeometry, and*

2. *p is 1-Mazoyer.*

Then, T has the strict order property.

Proof. We work in a finite slice of $\mathfrak{M}^{eq}$. Let $\mathfrak{M}'$ be the home sort of $\mathfrak{M}^{eq}$ together with the sort that contains the quotient of p by the equivalence relation $\sim$, where $x \sim y$ if and only if $\text{acl}(x) = \text{acl}(y)$. Since $\mathfrak{M}'$ and $\mathfrak{M}$ are bi-interpretable, $T' = \text{Th}(\mathfrak{M}')$ is still finitely axiomatizable, and T' has the strict order property if and only if T does. (In general, however, it seems that $\text{var}(T') > \text{var}(T)$, since the axioms for T' need to define the equivalence relation $\sim$ on p, requiring at least $(|B|+\beta(1))$-many variables.) Since each realization of $p/\sim$ is interalgebraic with a realization of p, it is easy to see that $p/\sim$ is 1-Mazoyer. Thus, T' satisfies the hypotheses of Theorem 4.4.2. □

In Example 2.2.8, we showed that if the lattice of algebraically closed sets is distributive, then T satisfies the hypothesis of Corollary 4.4.3. Thus, Corollary 4.4.3 implies Ivanov's theorem.

4.5 The tree property

In the proof of Theorem 4.4.2 in the previous section, the first use of the 1-Mazoyer hypothesis was to guarantee that $\text{acl}(XBC)$ did not cover $p(\mathfrak{M})$. In the following theorem, we use the assumption that p is stably embedded to achieve the same goal.

Theorem 4.5.1. *Let $T \in \mathcal{T}$. Suppose there is a 1-type p so that*

1. *algebraic closure restricted to p is a trivial geometry,*
2. *p is k-Mazoyer,*
3. *p is stably embedded.*

Let $j_0 = D^{\Delta_k}_{l_k}(x = x)$. Then, either, $j_0 = \infty$ or some formula in $\Delta_{(|B|+\text{var}(T)+j_0k)}$ has the strict order property.

(Recall that $l_k = \beta(k) + 1$.)

Proof. Suppose $j_0 \in \omega$ but no formula in $\Delta_{(|B|+\text{var}(T)+j_0k))}$ has the strict order property. As in the proof of Theorem 4.4.2, choose X so that X is defined over C, $X \subsetneq p(\mathfrak{M})$, X is strongly $(\text{var}(T) + j_0k)$-saturated with respect to p, and $p(\mathfrak{M}) \setminus X$ is infinite.

Since p is stably embedded, we may assume that $C \subset p(\mathfrak{M})$. Because algebraic closure restricted to p is a trivial geometry, $\operatorname{acl}(XCB) \cap p(\mathfrak{M}) = XC$. Thus, $p(\mathfrak{M}) \not\subseteq \operatorname{acl}(XCB)$.

Let E be a p-envelope of XBC. Then, as in Theorem 4.4.2, E is not a model, and thus there is a 1-type r over a small subset of E so that r is not realized in E.

An inductive argument identical to Lemma 4.3.1 shows that the strong $(\operatorname{var}(T) + j_0 k)$-saturation of X implies that $D_{l_k}^{\Delta_k}(r) > j_0$, contradicting the choice of j_0. □

Because the strict order property implies the tree property, we have the following result, which is Theorem 1.1.9.

Corollary 4.5.2. *If T satisfies the hypothesis of Theorem 4.5.1, then T has the tree property.*

Chapter 5

Groups involved in a theory

In this chapter, we prove that if $T \in \mathcal{T}$ is stable, then there are finitely many finite simple groups involved in T. Recall from the introduction that it is useful to think of the contrapositive of this result: if $T \in \mathcal{T}$ and there are infinitely many finite simple groups involved in T, then T has the order property (Theorem 1.1.6). The result appears in [14]; however, it was found independently by the present author. The proof we give here is very close to the proof in [14]. Both proofs adapt Hrushovski's argument that an uncountably categorical, finitely axiomatizable theory has finitely many simple groups involved [12]. In [14], Ivanov and Macpherson present various other hypotheses besides stability that are sufficient for the theorem; see Remark 5.2.4 below for a comment on their basic idea.

Unlike the previous chapter, we will not assume that T is ω-categorical in this chapter. Thus, A, B, etc. are sets with cardinality smaller than $|\mathfrak{M}|$, but they need not be finite.

5.1 Finite simple groups involved in a theory

In this section we define what it means for a finite simple group to be involved in a theory (Definition 5.1.2). The definition is due to Hrushovski [12], but is presented here in a different form. In Subsection 5.1.1, we show that Definition 5.1.2 is equivalent to Hrushovski's original definition.

We use the following standard notation: $\mathrm{Aut}(\mathfrak{M}/B)$ is the group of automorphisms of $\mathfrak{M}$ that fix B pointwise.

Definition 5.1.1. Let X be a subset of $\mathfrak{M}^n$.

1. We say that X is *normal over* B if X is invariant under the action of $\mathrm{Aut}(\mathfrak{M}/B)$. (So, X is a union of solution sets of complete n-types over B.)

2. When X is normal over B, $\mathrm{Aut}(X/B)$ denotes the set of permutations of X which extend to elements of $\mathrm{Aut}(\mathfrak{M}/B)$.

The following abuse of notation appears in the rest of the chapter: for any $\vec{a}$ and B, if X is the orbit of $\vec{a}$ over B, we write $\mathrm{Aut}(\vec{a}/B)$ for the group $\mathrm{Aut}(X/B)$. Note that if $\vec{a} \in \mathrm{acl}(B)$, then $\mathrm{Aut}(\vec{a}/B)$ is finite.

Definition 5.1.2. A finite simple group G is *involved in* T if there is a set B and a tuple $\vec{a} \in \mathrm{acl}(B)$ so that G is a composition factor of $Aut(\vec{a}/B)$.

Let $\mathcal{G} = \mathcal{G}(T)$ be the set of (isomorphism types of) finite simple groups involved in T. We say that T *has finitely many simple groups involved* if $\mathcal{G}$ is finite; otherwise, T *has infinitely many simple groups involved.*

The following fact seems to have been overlooked in [12] and [14].

Proposition 5.1.3. *In Definition 5.1.2, we may assume that B is finite.*

Proof. Take $\vec{a} \in \mathrm{acl}(B)$, where B is infinite. It suffices to show that there is a finite set $C \subset B$ so that $\vec{a} \in \mathrm{acl}(C)$ and $\mathrm{Aut}(\vec{a}/C) = \mathrm{Aut}(\vec{a}/B)$.

First, there is a finite set $B' \subset B$ so that $tp(\vec{a}/B')$ is equivalent to $tp(\vec{a}/B)$. To see this, choose a finite set $B_0 \subset B$ so that $\vec{a} \in \mathrm{acl}(B_0)$. If $tp(\vec{a}/B_0)$ is not equivalent to $tp(\vec{a}/B)$, then there is some formula over parameters $\vec{b}$ which witnesses the inequivalence, i.e. $tp(\vec{a}/B_0\vec{b})$ strictly implies $tp(\vec{a}/B_0)$. Let $B_1 = B_0\vec{b}$. If $tp(\vec{a}/B_1)$ is not equivalent to $tp(\vec{a}/B)$, we can find a finite set B_2 so that $tp(\vec{a}/B_2)$ strictly implies $tp(\vec{a}/B_1)$, as above. Iterating this process produces a descending chain of types below $tp(\vec{a}/B_0)$. Since $tp(\vec{a}/B_0)$ is algebraic, the process must stop, yielding a finite set B' so that $tp(\vec{a}/B')$ is equivalent to $tp(\vec{a}/B)$. Note that $\mathrm{Aut}(\vec{a}/B) \leq \mathrm{Aut}(\vec{a}/B')$.

Now, we produce a finite set C such that $B' \subseteq C \subseteq B$ and $\mathrm{Aut}(\vec{a}/B) = \mathrm{Aut}(\vec{a}/C)$. Let $X = \{\vec{a}_1, \ldots, \vec{a}_l\}$ be the set of realizations of $tp(\vec{a}/B')$. Suppose $\mathrm{Aut}(\vec{a}/B) < \mathrm{Aut}(\vec{a}/B')$. Then, there is a permutation σ of X so that the concatenated tuples $\vec{a}_1 \ldots \vec{a}_l$ and $\sigma(\vec{a}_1) \ldots \sigma(\vec{a}_l)$ have the same type over B' but different types over B. Thus, there is $\vec{b} \in B$ so that $\vec{a}_1 \ldots \vec{a}_l$ and $\sigma(\vec{a}_1) \ldots \sigma(\vec{a}_l)$ have different types over $B'\vec{b}$. Therefore, we have

$$\mathrm{Aut}(\vec{a}/B) \leq \mathrm{Aut}(\vec{a}/B'b) < \mathrm{Aut}(\vec{a}/B'),$$

where σ witnesses the strict inclusion. Now, if $\mathrm{Aut}(\vec{a}/B) < \mathrm{Aut}(\vec{a}/B'b)$, repeat the process described above. Since $\mathrm{Aut}(\vec{a}/B')$ is finite, we can only descend a finite number of times, and the desired set C exists.

□

5.1.1 The original definition

In this subsection, we give Hrushovski's original definition of a finite simple group involved in a theory, and show it is equivalent to Definition 5.1.2. First we introduce the notion of finite generation.

Definition 5.1.4. We say that X is *finitely generated over* B if there is $\vec{a}$ so that $X = \mathrm{dcl}(B\vec{a})$.

Notice that if X is finitely generated over B, then $|X| < |\mathfrak{M}|$.

Suppose that A is finitely generated and normal over B, say $A = \mathrm{dcl}(B\vec{a})$. Then, $\mathrm{Aut}(A/B) \simeq \mathrm{Aut}(\vec{a}/B)$.

Proposition 5.1.5. *If A is finitely generated and normal over B, then $A \subseteq \mathrm{acl}(B)$.*

Proof. Say $A = \mathrm{dcl}(B\vec{a})$, where the length of $\vec{a}$ is n. Note that $\vec{a} \in A^n$, and by normality, the orbit of $\vec{a}$ under $\mathrm{Aut}(\mathfrak{M}/B)$ is contained in A^n. Thus, the orbit of $\vec{a}$ has cardinality less than $|\mathfrak{M}|$. Since the orbit is type definable, bounded size means finite, i.e. $\vec{a} \in \mathrm{acl}(B)$. Hence, $A \subseteq \mathrm{acl}(B\vec{a})$. □

By the proposition, if A is finitely generated and normal over B, then $\mathrm{Aut}(A/B)$ is finite.

We can now state Hrushovski's original definition.

Definition 5.1.6 (Hrushovski [12]). A finite simple group G is *involved in* T if there are sets A and B so that A is both normal and finitely generated over B, and G is a composition factor of $\mathrm{Aut}(A/B)$.

Proposition 5.1.7. *Definitions 5.1.6 and 5.1.2 are equivalent.*

Proof. Suppose that G is involved via Definition 5.1.2, i.e. G is a composition factor of $\mathrm{Aut}(\vec{a}/B)$, where $\vec{a} \in \mathrm{acl}(B)$. Let $\{\vec{a}_0, \ldots \vec{a}_n\}$ be the set of realizations of $tp(\vec{a}/B)$. Then, letting $A = \mathrm{dcl}(B\vec{a}_0, \ldots, \vec{a}_n)$, we have that A is normal and finitely generated over B, with $\mathrm{Aut}(A/B) = \mathrm{Aut}(\{\vec{a}_0, \ldots, \vec{a}_n\}/B) = \mathrm{Aut}(\vec{a}/B)$. (The last equality is our notational convention.) This shows that G is involved in T in the sense of Definition 5.1.6.

Suppose that G is involved in T according to Definition 5.1.6; i.e. G is a composition factor of $\mathrm{Aut}(A/B)$, where A is normal and finitely generated over B. Let $A = \mathrm{dcl}(B\vec{a})$. Then, $\mathrm{Aut}(A/B) = \mathrm{Aut}(\vec{a}/B)$ and by Proposition 5.1.5, $\vec{a} \in \mathrm{acl}(B)$. Thus, G is involved in the sense of Definition 5.1.2. $\square$

Hrushovski introduced the definition of groups involved in a theory to show that finitely axiomatizable, uncountably categorical theories are locally modular. His proof hinges on the following result.

Theorem 5.1.8 (Hrushovski [12] Theorem A). *If T is stable and some minimal set in $\mathfrak{M}$ is not locally modular, then T has infinitely many groups involved.*

5.2 Groups involved in a finitely axiomatizable ω-categorical theory

In this section, we prove that if $T \in \mathcal{T}$ is stable, then $\mathcal{G}(T)$ is finite. We assume that T is ω-categorical, so the sets A, B, etc. are finite. By Proposition 5.1.3, this convention does not clash with the definition $\mathcal{G}(T)$.

We will need the following easy group-theoretic lemma.

Lemma 5.2.1. *Suppose that a group G has a chain of subgroups $G = G_0 \geq G_1 \geq \cdots \geq G_n = 1$, where $[G_i : G_{i+1}] \leq m$ for each i. Then, every composition factor of G has cardinality at most $m!$.*

Proof. We do induction on n. If $n = 0$, there is nothing to prove.

Assume $n > 0$. Let G/G_1 denote the set of left cosets of G_1; by hypothesis, $|G/G_1| \leq m$. Let G act on G/G_1 by left multiplication, and let K be the kernel of this action. Then, $K \leq G_1$, and $[G : K] \leq m!$. For each $i > 0$, let $K_i = K \cap G_i$. Thus, $K_1 = K$ and $G \trianglerighteq K_1 \geq \cdots \geq K_n = 1$. Note that $[K_i : K_{i+1}] = [K \cap G_i : K \cap G_{i+1}] \leq m$. Applying induction to the chain of K_i's, we have that all composition factors of K_1 have size at most $m!$.

But a composition factor of G is either a composition factor of K_1 or else a composition factor of G/K_1. Since $|G/K_1| \leq m!$, its composition factors have size at most $m!$, and we are done. □

Recall that a *small set* is one of size less than $\mathrm{var}(T)$, and a *small type* is one whose domain is a small set.

Theorem 5.2.2. *Suppose that $T \in \mathcal{T}$ and T is stable. Further suppose that there is $m \in \omega$ such that for any small 1-type r, the multiplicity of r is at most m. Then, for all $G \in \mathcal{G}(T)$, $|G| \leq m!$. A fortiori, $\mathcal{G}(T)$ is finite.*

Proof. Take $\vec{a} \in \mathrm{acl}(B)$. Let $\{\vec{a}_0, \ldots, \vec{a}_l\}$ be the set of realizations of $tp(\vec{a}/B)$. We construct a descending chain of subgroups in $G = \mathrm{Aut}(\vec{a}/B)$ that terminates at 1, where each step of the chain has bounded index.

As discussed in Section 4.1, we can build a model over B as follows. Build an increasing chain $B = B_0 \subset B_1 \subset \cdots \subset B_i \subset \cdots$ so that

1. $B_{i+1} = B_i d_i$, where d_i realizes some small type $r_i \in S_1(C_i)$, with $C_i \subseteq B_i$;

2. $N = \bigcup_i B_i$ is an elementary substructure of $\mathfrak{M}$.

Furthermore, we choose each d_i so that d_i is independent from B_i over C_i.

Let $G_i = \text{Aut}(\vec{a}/B_i)$. Since $N \preccurlyeq \mathfrak{M}$ and $\vec{a} \in \text{acl}(B)$, we have that $\vec{a} \in N$. Thus, there is an n so that $\vec{a} \in B_n$ and $\text{Aut}(\vec{a}/B_n) = 1$. So we have the chain

$$G = G_0 \geq G_1 \geq \cdots \geq G_n = 1.$$

Claim 5.2.3. *For each i with $0 \leq i < n$, we have $[G_i : G_{i+1}] \leq m$.*

Proof. Recalling the definition of the G_i's, we see that $[G_i : G_{i+1}] = [\text{Aut}(\vec{a}/B_i) : \text{Aut}(\vec{a}/B_i d_i)]$, where $d_i \models r_i(x) \in S_1(C_i)$ and $|C_i| < \text{var}(T)$. In order to cut down on subscripts, let $d = d_i$, $r = r_i$, $C = C_i$.

Let G_i act on $S_1(\vec{a}_0, \ldots, \vec{a}_l B_i)$ (considered as a set of syntactic objects). Let $\mathcal{O}$ be the orbit of $q(x) = tp(d/\vec{a}_0, \ldots, \vec{a}_l B_i)$ under this action. We show that $G_{i+1} = \text{Aut}(\vec{a}/B_i d)$ is the stabilizer of q. It is clear that G_{i+1} is contained in the stabilizer. Conversely, suppose that $\sigma \in G_i$ stabilizes q. By the definition of G_i, σ is a permutation of $\{\vec{a}_0, \ldots, \vec{a}_l\}$ which extends to an automorphism $\tilde{\sigma} \in \text{Aut}(\mathfrak{M}/B_i)$. Note that $\tilde{\sigma}(q) = \sigma(q) = q$. Thus, $\tilde{\sigma}(d)$ realizes q, and there is $\tau \in Aut(\mathfrak{M}/\vec{a}_0, \ldots, \vec{a}_l B_i)$ so that $\tau(\tilde{\sigma}(d)) = d$. Hence, $\tau \circ \tilde{\sigma} \in \text{Aut}(\mathfrak{M}/B_i d)$, and since $\tau \circ \tilde{\sigma} \restriction \{\vec{a}_0, \ldots, \vec{a}_l\} = \sigma$, we have shown that $\sigma \in G_{i+1}$. Thus, G_{i+1} is the stabilizer of q.

Therefore, we know that $[G_i : G_{i+1}] = |\mathcal{O}|$. Now, $|\mathcal{O}|$ is bounded by the number of extensions of $tp(d/B_i)$ to $\{\vec{a}_0, \ldots, \vec{a}_l\} \cup B_i$. Since $\{\vec{a}_0, \ldots, \vec{a}_l\} \subseteq \text{acl}(B_i)$, each of these extensions is a non-forking extension, so their number is at most $\text{mult}(d/B_i)$. Since d was chosen to be free from B_i over C, $\text{mult}(d/B_i) \leq \text{mult}(d/C) \leq m$. Therefore, $[G_i : G_{i+1}] \leq m$, as claimed. □

Applying Lemma 5.2.1, we see that every composition factor of G has size at most $m!$. Therefore, any group involved in T has size at most $m!$. This completes the proof of the lemma. □

Remark 5.2.4. Much less than stability is needed in the proof of the lemma. All we really need is some notion of a "canonical" extension of types (not necessarily non-forking) that satisfies the following properties.

1. Let $r \in S_1(C)$ be a small type. Then, for any $B \supseteq C$, there is a canonical extension of r in $S_1(B)$.

2. Each extension of a type to the algebraic closure of its domain is canonical.

3. If p_1 is a canonical extension of p_0 and p_2 is a canonical extension of p_1, then p_2 is a canonical extension of p_0.

4. There is a bound on the number of canonical extensions of a small type to any set.

When building the model, choose each d_i to realize a canonical extension of r_i to B_i. Then, the properties listed above allow the proof to go through as before.

The idea of such canonical extensions (called *strongly determined extensions*) is investigated more fully in [14].

Corollary 5.2.5 (Ivanov & Macpherson, [14]). *If $T \in \mathcal{T}$ and T is stable, then $\mathcal{G}(T)$ is finite.*

Proof. We need only show that there is an $m \in \omega$ bounding the multiplicity of every small type. Take $r \in S_1(C)$, with $|C| < \text{var}(T)$. By the Finite Equivalence Relation Theorem, the multiplicity of r is the same as the number of strong types over C which extend r. But, by ω-categoricity, there is a uniform bound on the number of finite equivalence relations definable over C, and a uniform bound on the number of classes in such an equivalence relation. Hence, there is a uniform bound on the number of strong types. $\square$

Remark 5.2.6. Let T be an ω-stable, ω-categorical theory. The proof of the non-finite axiomatizability of T breaks into two parts. First, one proves that all strongly minimal sets in $\mathfrak{M}$ are locally modular (the "Zilber dichotomy"). Then, a coordinatization analysis allows one to prove the finite substructure property for T. Historically, the first step has been proved by the machinery of geometric stability theory (Zilber [31], Hrushovski [11]), or by the classification of finite simple groups (Cherlin and Mills [7, Appendix]). If one is not interested in the full structure theory of T, but is interested only in the fact that T is not finitely axiomatizable, then Corollary 5.2.5 provides a way to avoid having to prove the Zilber dichotomy. Start by assuming that T *is* finitely axiomatizable. By the corollary, $\mathcal{G}(T)$ is finite. Then, by Theorem 5.1.8 (i.e. Hrushovski's Theorem A from [12]), every strongly minimal set in $\mathfrak{M}$ is locally modular. Now, deduce the finite substructure property as in [7], contradicting our assumption that T is finitely axiomatizable.

Whereas this method avoids some of the complexities involved in the proof of the Zilber dichotomy, the proof of Theorem 5.1.8 has its own complexities. Hrushovski's proof of Theorem 5.1.8 uses the fact that a strongly minimal set that is not locally modular is not k-pseudolinear for any k [11]. The proof of this fact found in [11] is actually rather close to a proof of the Zilber dichotomy. However, it is possible to prove Theorem 5.1.8 via Buechler's result that pseudomodular strongly minimal sets are locally modular (Theorem 2.2.13 above), and proving this result does seem to be different than proving the Zilber dichotomy.

We close the chapter with the following result, mentioned in the introduction, that complements Corollary 5.2.5.

Proposition 5.2.7. *Let T be ω-categorical. If T has bounded algebraic arity, then $\mathcal{G}(T)$ is finite.*

Proof. Assume that T has k-ary algebraic closure. Take $\vec{a} \in \mathrm{acl}(B)$. Let $\{\vec{a}_1, \ldots, \vec{a}_l\}$ be the set of realizations of $tp(\vec{a}/B)$. Let $G = \mathrm{Aut}(\vec{a}/B)$. The proof is much the same as the proof for Theorem 5.2.2. As before, we build a chain of subgroups of G that has small index at each step and terminates at 1; however, instead of constructing a model over B, we simply construct $\mathrm{acl}(B)$.

By Proposition 2.1.9, we know that $\mathrm{acl}_k^\omega() = \mathrm{acl}()$. Thus, we can build $\mathrm{acl}(B)$ as a union of a chain, where at each stage we realize an algebraic type over a k-element subset. That is, we construct an increasing chain $B = B_0 \subset B_1 \subset \cdots \subset B_i \subset \cdots$ so that

1. $B_{i+1} = B_i d_i$, where d_i realizes some algebraic type $r_i \in S_1(C_i)$, with $C_i \subseteq B_i$ and $|C_i| < k$;

2. $\bigcup_i B_i = \mathrm{acl}(B)$.

Let $G_i = \mathrm{Aut}(\vec{a}/B_i)$. Then, since $\vec{a}$ appears at some stage in the construction, there is n so that $G_n = 1$. Thus, we have the chain

$$G = G_0 \geq G_1 \geq \cdots \geq G_n = 1.$$

As in the proof of Theorem 5.2.2, let G_i act on $S_1(\{\vec{a}_1, \ldots, \vec{a}_n\} \cup B)$. Exactly as before, G_{i+1} is the stabilizer of $q(x) = tp(d_i/\{\vec{a}_1, \ldots, \vec{a}_n\} \cup B)$. Thus, the index $[G_i : G_{i+1}]$ is equal to the size of the orbit of q under the action. Every type in the orbit is an extension of $tp(d_i/B_i)$ to $\{\vec{a}_1, \ldots, \vec{a}_n\} \cup B$, and there are at most $\mathrm{mult}(d_i/B_i)$ such extensions. (Here, mult() is algebraic multiplicity, i.e. the number of realizations of the algebraic type.) Now, $\mathrm{mult}(d_i/B_i) \leq \mathrm{mult}(d_i/C_i)$, and there is a uniform bound on the size of $\mathrm{mult}(d_i/C_i)$, since $|C_i| < k$. Applying Lemma 5.2.1, we are done. □

Chapter 6

Further questions

Below, we list some questions for further investigation. Of course, Macpherson's conjecture and the order property conjecture remain open, and are the ultimate goals.

6.1 A conjecture on the independence property

The following conjecture reduces Macpherson's conjecture to the order property conjecture.

Conjecture 6.1.1. If $T \in \mathcal{T}$ has the independence property, then it has the strict order property. Equivalently, if $T \in \mathcal{T}$ is unstable, then it has the strict order property.

There are theories in $\mathcal{T}$ with the independence property, such as the theory of the Fraïssé limit of partial orders. However, in the examples, the independence property quite clearly comes from an ordering. Conjecture 6.1.1 seems reasonable in the sense that it is hard to imagine how, in the absence of order, a finite set of axioms would guarantee the independence property. This line of reasoning is not specific to the ω-categorical case, so we ask the following question.

Question 6.1.2. Is there any finitely axiomatizable complete theory with the independence property but not the strict order property?

Peretyatkin [18] shows that given a recursively axiomatizable theory T, it is possible to produce an associated finitely axiomatizable theory T' that shares many properties with T, for example uncountable categoricity and λ-stability (but not ω-categoricity!). A careful analysis of Peretyatkin's work might provide insight into Question 6.1.2.

6.2 Questions relating to Chapter 4

If one is looking to strengthen the results of this dissertation, the following question is an obvious one.

Question 6.2.1. Does $T \in \mathcal{T}$ necessarily have bounded algebraic arity? If not, does it necessarily have a type satisfying the Mazoyer hypothesis?

More generally, does *every* ω-categorical theory have bounded algebraic arity (or a type satisfying the Mazoyer hypothesis)? It seems reasonable to narrow this question at first by considering, for example, stable or simple theories.

Another direction for improvement is to weaken the hypotheses necessary for the strict order property. For example, we ask

Question 6.2.2. If $T \in \mathcal{T}$ has a modular lattice of algebraically closed sets, can we prove that T has the strict order property?

6.3 Questions about examples

There is no general method for constructing interesting examples in $\mathcal{T}$; to find such a construction technique would be highly desirable. Even if the technique produces theories with the strict order property and bounded algebraic arity (and so is not useful for answering the questions raised above), there are still plenty of other basic questions it might answer, for example, the following question raised in the introduction.

Question 6.3.1. Is there an interesting example in $\mathcal{T}$ that satisfies the hypotheses of Theorem 1.1.8? By "interesting", I mean the example should not satisfy the hypotheses because it, or a piece of it, has a distributive lattice of closed sets.

A related question is

Question 6.3.2. Is there a theory in $\mathcal{T}$ with a lattice of closed sets that is modular but not distributive? (An obvious candidate is the theory of atomless Boolean algebra; is it modular?)

A first step toward a construction method might be a technique for constructing theories in $\mathcal{T}'$ (the class of theories finitely axiomatized modulo a universal theory). It would seem that constructing theories in $\mathcal{T}'$ would be easier than constructing theories in $\mathcal{T}$; however, as mentioned in Section 1.3.4, I do not know of any examples in $\mathcal{T}' \setminus \mathcal{T}$.

Question 6.3.3. Does $\mathcal{T}' = \mathcal{T}$?

It seems quite unlikely that the answer to this question is yes. There might even be an example with quantifier elimination, i.e. a Fraïssé limit with a finitely axiomatizable embedding property, but a non-finitely axiomatizable class of substructures.

The partial orders of finite width, discussed in Chapter 3, are a potential source of interesting examples.

Question 6.3.4. Is there an ω-categorical partial order of finite width that is not nuclear? If so, what sort of algebraic closure operator does it have?

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