

# KNESER'S PROOF OF THE FOUR VERTEX THEOREM

CLAY SHONKWILER

Suppose  $C$  is a plane curve violating the Four Vertex Theorem. Then there are points  $a, b \in C$  such that  $a$  has maximum radius of curvature,  $b$  has minimum radius of curvature and the radii of curvature monotonically decrease from  $a$  to  $b$ . Consider the evolute  $E$  to  $C$  and points  $a_1, b_1 \in E$  corresponding to  $a$  and  $b$ . Since  $E$  contains no straight segments and the arc length from  $a_1$  to  $b_1$  is equal to the difference of the radii of curvature, it is easy to see that each point on  $C$  except possibly  $a$  and  $b$  lies on exactly two osculating circles.

To see that this is impossible, stereographically project  $C$  to a sphere tangent to the plane of  $C$ ; let  $C_0$  be the image of  $C$ .<sup>1</sup> Note that stereographic projection takes osculating circles to osculating circles. Also, since  $C_0$  is closed, it intersects any plane in an even number of points (counting multiplicity).

If we stereographically project  $C_0$  from a point  $p \in C_0$  to a plane  $P_2$  parallel to  $T_p S^2$ , let  $C_{01}$  be the image of  $C_0$ . If  $P_1$  is any plane not tangent to  $C_0$  and passing through  $p$ , then  $(P_1 \cap C_0) \setminus \{p\}$  consists of an odd number of points; the images of those points on  $C_{01}$  must lie on a straight line. Since the choice of  $P_1$  was (mostly) arbitrary, we see that  $C_{01}$  intersects any straight line in an odd number of points, so  $C_{01}$  is an “odd circuit”.

Now, a theorem of Möbius guarantees that any odd circuit has at least three inflection points. It's easy to see that an inflection point on  $C_{01}$  corresponds to a point on  $C_0$  whose osculating circle passes through  $p$ . Since the choice of  $p$  was arbitrary, this, in turn, implies that every point on the original curve  $C$  lies on at least four osculating circles, which contradicts the result in the first paragraph. Thus, there is no curve  $C$  violating the Four Vertex Theorem.

DRL 3E3A, UNIVERSITY OF PENNSYLVANIA  
*E-mail address:* shonkwil@math.upenn.edu

---

<sup>1</sup>Although Kneser doesn't explicitly say this, the above paragraph has problems if  $C$  has both positive and negative curvatures. In that case, stereographically project to the sphere first and the above argument, with only minor modifications, shows that each point except possibly the max and the min lies on exactly two osculating circles