Math 336 Midterm

Due 5:00 PM Tuesday, March 23.

Name: ____

You may refer to your notes, your text, or your homework. Do not discuss the questions with anyone except me.

There is no time limit, and the exam is due at 5:00 PM on Tuesday, March 23. If you plan to ask me questions about course content, please do not open the test until after you do so.

Please do not sign the following until the examination is completed.

I accept full responsibility under the Haverford Honor Code for my conduct on this examination.

Signed: ____

1. (8 points) Construct a Δ-complex K with the following simplicial homology:

$$H_n^{\Delta}(K) = \begin{cases} \mathbb{Z}^2 & n = 2 \\ \mathbb{Z} \oplus \mathbb{Z}/3 & n = 1 \\ \mathbb{Z} & n = 0 \\ 0 & \text{otherwise.} \end{cases}$$

   Of course, you need to prove that your K actually has that homology.

2. (15 points) Given a chain map $f : A_* \to B_*$, we can form a new chain complex called the **mapping cone** of f, denoted $(C(f)_*, \partial_f)$, as follows: the chain groups are defined by

$$C(f)_n = A_{n-1} \oplus B_n$$

   and the boundary map is defined by:

$$\partial_f(a, b) = (-\partial_A a, f(a) + \partial_B b).$$

   (a) Prove that $(C(f)_*, \partial_f)$ is a chain complex.

   (b) Prove that there exists a long exact sequence of the form

$$\cdots \to H_n(B) \to H_n(C(f)) \to H_{n-1}(A) \xrightarrow{\partial_*} H_{n-1}(B) \to \cdots.$$

   NOTE: Yes, the dimensions are correct! In this long exact sequence, the connecting homomorphism is between $H_n(A)$ and $H_n(B)$.

   HINT: You might want to consider the chain complex A_*^-, which has the same chain groups as A_*, but whose differential is $\partial_A^- = -\partial_A$. Does this have the same homology as A_*?

   (c) Show that the connecting homomorphism in the long exact sequence above is equal to f_*.

   (d) Prove that f_* is an isomorphism if and only if the mapping cone has trivial homology.

   NOTE: You might wonder where the name "mapping cone" comes from. Here's a hint: it's not hard to prove that if $g : K \to K$ is the identity map from a Δ-complex to itself, then $C_*(g_\#) \simeq \Delta_*(CK, \{v'\})$, with notation as in HW#2, problem #3. Part (d) then solves that problem immediately.

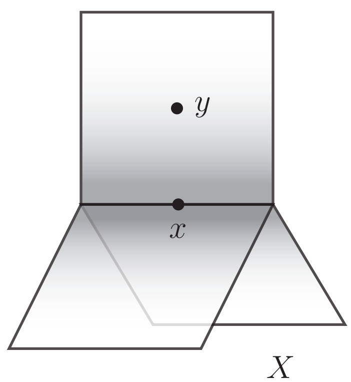
3. (12 points) Given a space X, let SX be the *suspension of* X, defined by $SX = X \times I/\sim$, where the equivalence relation collapses $X \times \{0\}$ to a point x_0 and $X \times \{1\}$ to another point x_1; see also p. 8 of Hatcher. Prove that:

$$\tilde{H}_n(X) \simeq \tilde{H}_{n+1}(SX).$$

4. (15 points) Give examples of the following (and show that they are indeed examples!), or prove that no such example exists:

   (a) Two pairs (X, A) and (Y, B) such that X and Y are homotopy equivalent, A and B are homotopy equivalent, but $H_n(X, A) \neq H_n(Y, B)$. A simplicial calculation is OK here.

   (b) Let X be a space constructed out of three squares like so:



   Give an example of a homeomorphism $f : X \to X$ that sends x to y.

   (c) Let K be a Δ-complex such that $|K| \simeq T^2$, and suppose that $H_1^{\Delta}(K) = \langle [a], [b] \rangle$. Find a Δ-map $f : K \to K$ such that $f_*[a] = [b]$ and $f_*[b] = [a]$.