Math 215 Final Exam Practice Problems

1. For each of the following statements, say whether it is true or false. If the statement is false, give a counterexample (no need to give a proof if true).

   (a) If eigenvectors $\vec{v}$ and $\vec{w}$ correspond to distinct eigenvalues, then $\vec{v}^T\vec{w} = 0$.

   Answer: False. Consider the matrix $A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$. The eigenvalues of A are $\lambda_1 = 1$ and $\lambda_2 = 2$, with corresponding eigenvectors $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Since

   $$\vec{v}_1^T\vec{v}_2 = [1\ 0]\begin{bmatrix} 1 \\ 1 \end{bmatrix} = 1 \neq 0,$$

   A provides a counterexample to the statement.

   (b) Let A be an $m \times n$ matrix and let $\vec{b}$ be a vector in $\mathbb{R}^m$. If $m < n$, then $A\vec{x} = \vec{b}$ has infinitely many solutions.

   Answer: False. Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \end{bmatrix}$ and let $\vec{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Then the equation

   $$A\vec{x} = \vec{b}$$

   has *no* solutions.

   (c) If A is an $m \times n$ real matrix, then the nullspace of A^T is the orthogonal complement of the column space of A.

   Answer: True.

   (d) If S and T are subspaces of $\mathbb{R}^2$, then their union (i.e., the set of vectors which are in either S or T) is also a subspace of $\mathbb{R}^2$.

   Answer: False. Let S be the line containing the vector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and let T be the line containing the vector $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Then S and T are certainly subspaces of $\mathbb{R}^2$, but their union $S \cup T$ is not a subspace because it is not closed under addition. To see this, notice that $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ are in $S \cup T$, but

   $$\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

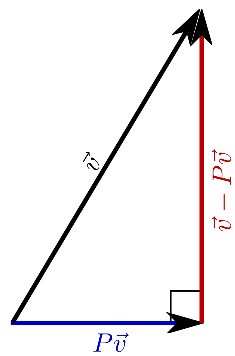
   is in neither S nor T, and so is not in $S \cup T$.

   (e) Let S be a plane through the origin in $\mathbb{R}^3$ and let P be the matrix which projects onto the plane S. Then for any $\vec{v} \in \mathbb{R}^3$,

   $$\|\vec{v}\|^2 = \|P\vec{v}\|^2 + \|\vec{v} - P\vec{v}\|^2.$$

   Answer: True. If P is the projection onto the plane S, then $\vec{v} - P\vec{v}$ is perpendicular to $P\vec{v}$. Therefore, the Pythagorean theorem applied to the right triangle pictured at the top of the next page gives that

   $$\|\vec{v}\|^2 = \|P\vec{v}\|^2 + \|\vec{v} - P\vec{v}\|^2.$$



2. Is the set of all orthogonal $n \times n$ real matrices a vector space? What about the set of all skew-Hermitian $m \times m$ complex matrices?

   Answer: No. Notice that
   $$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$
   are both orthogonal matrices, but their sum,
   $$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$
   has determinant 0 and so cannot be orthogonal (since all orthogonal matrices have determinant 1).

3. Suppose the first row of A is $2, 3$ and its eigenvalues are $i, -i$. Find A.

   Answer: Let
   $$A = \begin{bmatrix} 2 & 3 \\ c & d \end{bmatrix}.$$
   Then the trace of A is $2 + d$. On the other hand, the trace of A is the sum of the eigenvalues of A, so the trace equals $i + (-i) = 0$. Therefore,
   $$2 + d = 0,$$
   so $d = -2$.

   Also, the determinant of A is equal, on the one hand, to $i(-i) = 1$ and, on the other hand,
   $$2(-2) - 3c = -4 - 3c.$$
   Hence, $-4 - 3c = 1$ and so $c = -5/3$. Therefore,
   $$A = \begin{bmatrix} 2 & 3 \\ -5/3 & -2 \end{bmatrix}.$$

4. If $\vec{x}_1, \vec{x}_2$ are the columns of S, what are the eigenvalues and eigenvectors of
   $$A = S \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} S^{-1} \quad \text{and} \quad B = S \begin{bmatrix} 3 & 4 \\ 0 & 1 \end{bmatrix} S^{-1}?$$

   Answer: Since A has already been diagonalized, we see that the eigenvalues of A are 3 and 1 and that the eigenvectors are the columns of S, namely $\vec{x}_1$ and $\vec{x}_2$.

   Since B is similar to the upper triangular matrix $\begin{bmatrix} 3 & 4 \\ 0 & 1 \end{bmatrix}$ and since the eigenvalues of the triangular matrix are its diagonal entries, we see that the eigenvalues of B are also 3 and 1. To determine the eigenvectors of B, we first find the eigenvectors of $\begin{bmatrix} 3 & 4 \\ 0 & 1 \end{bmatrix}$.

The eigenvector corresponding to the eigenvalue 3 is in the nullspace of

$$B - 3I = \begin{bmatrix} 0 & 4 \\ 0 & -2 \end{bmatrix},$$

so we see that this eigenvector is $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

The eigenvector corresponding to the eigenvalue 1 is in the nullspace of

$$B - 1I = \begin{bmatrix} 2 & 4 \\ 0 & 0 \end{bmatrix},$$

so this eigenvector is $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$.

Therefore,

$$\begin{bmatrix} 3 & 4 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix},$$

which means that we can diagonalize B as

$$B = S \begin{bmatrix} 3 & 4 \\ 0 & 1 \end{bmatrix} S^{-1} = S \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} S^{-1} = \left(S \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \left(\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} S^{-1} \right).$$

Therefore, the eigenvectors of B are the columns of

$$S \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} = [\vec{x}_1 \ \vec{x}_2] \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} = [\vec{x}_1 \quad (-2\vec{x}_1 + \vec{x}_2)].$$

5. If A is an $n \times (n-1)$ matrix with rank $n-2$ and $\vec{b} \in \mathbb{R}^n$ such that $A\vec{v} = \vec{b}$ for some $\vec{v} \in \mathbb{R}^{n-1}$, what is the dimension of the space of *all* solutions to the equation $A\vec{x} = \vec{b}$?

   Answer: Since A has rank $n-2$, we see that

   $$n - 1 = \dim \operatorname{col}(A) + \dim \operatorname{nul}(A) = \operatorname{rank}(A) + \dim \operatorname{nul}(A) = n - 2 + \dim \operatorname{nul}(A),$$

   so the nullspace of A is 1-dimensional. Therefore, the space of solutions of $A\vec{x} = \vec{b}$ is also 1-dimensional.

6. (a) Prove that the eigenvalues of A are the eigenvalues of A^T.

   Proof. The eigenvalues of A are the roots of $\det(A - \lambda I)$, whereas the eigenvalues of A^T are the roots of $\det(A^T - \lambda I)$. However, since $I^T = I$, we see that

   $$(A - \lambda I)^T = A^T - \lambda I^T = A^T - \lambda I.$$

   Therefore,

   $$\det(A - \lambda I) = \det\left((A - \lambda I)^T\right) = \det(A^T - \lambda I).$$

   since the polynomials are the same, their roots are of course the same, and so the eigenvalues of A and A^T are equal. □

   (b) If A and B are $n \times n$ real symmetric matrices, show that AB and BA have the same eigenvalues. [Hint: Use part (a)]

   Proof. By part (a), AB and $(AB)^T = B^T A^T$ have the same eigenvalues. However, since both A and B are symmetric,

   $$B^T A^T = BA,$$

   so we can conclude that AB and BA have the same eigenvalues. □

7. Is there a real 2×2 matrix $A \neq I$ such that $A^3 = I$?

   Answer: Yes. We can simply let A be the matrix of the linear transformation which rotates the plane by $2\pi/3$, since performing this rotation three times has the same effect as doing nothing.

   In particular,

$$A = \begin{bmatrix} \cos(2\pi/3) & -\sin(2\pi/3) \\ \sin(2\pi/3) & \cos(2\pi/3) \end{bmatrix} = \begin{bmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{bmatrix},$$

   and you can easily check that, indeed, $A^3 = I$.

8. Suppose A is diagonalizable and that we define the polynomial

$$p(\lambda) = \det(A - \lambda I) = c_n\lambda^n + c_{n-1}\lambda^{n-1} + \ldots + c_1\lambda + c_0.$$

   Show that

$$p(A) = c_n A^n + c_{n-1}A^{n-1} + \ldots + c_1 A + c_0 I$$

   is the zero matrix.

   NOTE: The fact that this is true for *any* matrix (regardless of whether it's diagonalizable) is called the *Cayley-Hamilton Theorem*.

   Proof. Since A is diagonalizable,

$$A = S\Lambda S^{-1},$$

   where the entries of Λ are the eigenvalues λ_i of A. Therefore,

$$\begin{aligned} p(A) &= c_n A^n + c_{n-1}A^{n-1} + \ldots + c_1 A + c_0 I \\ &= c_n S\Lambda^n S^{-1} + c_{n-1}S\Lambda^{n-1}S^{-1} + \ldots + c_1 S\Lambda S + c_0 SS^{-1} \\ &= S(c_n\Lambda^n + c_{n-1}\Lambda^{-1} + \ldots + c_1\Lambda + c_0 I)S^{-1}. \end{aligned}$$

   Now, the expression inside the parentheses is a diagonal matrix whose entries are

$$p(\lambda_i) = c_n\lambda_i^n + c_{n-1}\lambda_i^{n-1} + \ldots + c_1\lambda_i + c_0.$$

   Recall that the eigenvalues of A are the roots of the polynomial $p(\lambda)$, so each of these terms is zero. Therefore,

$$p(A) = S0S^{-1},$$

   which is the zero matrix □

9. Suppose $A = \vec{u}\vec{v}^T$ where $\vec{u}, \vec{v} \in \mathbb{R}^n$ and $\vec{v} \neq \vec{0}$.

   (a) What is the rank of A?

   Answer: Since each row of A is just a multiple of $\vec{v}^T$, we see that the row space of A has dimension 1. Since the dimension of the row space equals the rank of A, this implies that A has rank 1.

   (b) Show that $\vec{u}$ is an eigenvector of A. What is the corresponding eigenvalue?

   Answer: Clearly,

$$A\vec{u} = (\vec{u}\vec{v}^T)\vec{u} = \vec{u}(\vec{v}^T\vec{u}) = \langle \vec{v}, \vec{u} \rangle \vec{u},$$

   so $\vec{u}$ is an eigenvector with eigenvalues $\langle \vec{u}, \vec{v} \rangle$.

(c) What are the other eigenvalues of A?

Answer: Since A has rank 1, the nullspace of A has dimension $n-1$, so the only other eigenvalue of A is 0. Note that, if $\vec{w}$ is perpendicular to $\vec{v}$, then

$$A\vec{w} = (\vec{u}\vec{v}^T)\vec{w} = \vec{u}(\vec{v}^T\vec{w}) = \vec{u}\langle\vec{v},\vec{w}\rangle = 0,$$

so the vectors perpendicular to $\vec{v}$ are all eigenvectors corresponding to the eigenvalue 0.

(d) Compute the trace of A in two different ways.

Answer: We know that the trace of A is the sum of the eigenvalues, so

$$\text{trace}(A) = \langle\vec{u},\vec{v}\rangle + 0 \ldots + 0 = \langle\vec{u},\vec{v}\rangle.$$

On the other hand,

$$A = \vec{u}\vec{v}^T = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix} = \begin{bmatrix} u_1v_1 & u_1v_2 & \cdots & u_1v_n \\ u_2v_1 & u_2v_2 & \cdots & u_2v_n \\ \vdots & \vdots & \ddots & \vdots \\ u_nv_n & u_nv_2 & \cdots & u_nv_n \end{bmatrix}.$$

Therefore, by definition,

$$\text{trace}(A) = u_1v_1 + u_2v_2 + \ldots + u_nv_n = \langle\vec{u},\vec{v}\rangle,$$

agreeing with our earlier answer.

10. Let V be an n-dimensional vector space and suppose $T : V \to V$ is a linear transformation such that the range of T is equal to the set of vectors that T sends to $\vec{0}$. In other words,

$$\{T(\vec{v}) : \vec{v} \in V\} = \{\vec{v} \in V : T(\vec{v}) = \vec{0}\}.$$

(a) Show that n must be even.

Proof. Let A be the matrix representation of T. Then the range of T is the column space of A and the set of vectors that T sends to $\vec{0}$ is the nullspace of A. Since the two are equal, they have the same dimension r. Also,

$$n = \dim \text{col}(A) + \dim \text{nul}(A) = r + r = 2r,$$

so n is even. □

(b) Give an example of such a T.

Answer: Let T be the transformation given by multiplying by the matrix

$$A = \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}.$$

Then A row-reduces to $\begin{bmatrix} -1 & -1 \\ 0 & 0 \end{bmatrix}$, so the nullspace of A consists of all multiples of $\begin{bmatrix} -1 & 1 \end{bmatrix}$. Since this vector is also a copy of both of the columns, the column space of A also consists of multiples of this vector, so this choice of T works to give an example.

11. Find the LU decomposition of the matrix

$$A = \begin{bmatrix} 3 & -1 & 2 \\ -3 & -2 & 10 \\ 9 & -5 & 6 \end{bmatrix}.$$

Answer: We need to do elimination to get the upper triangular matrix U. First, add row 1 to row 2 and subtract three times row 1 from row 3, yielding:

$$\begin{bmatrix} 3 & -1 & 2 \\ 0 & -3 & 12 \\ 0 & -2 & 0 \end{bmatrix}.$$

Next, subtract 2/3 of row 2 from row 3 to get

$$U = \begin{bmatrix} 3 & -1 & 2 \\ 0 & -3 & 12 \\ 0 & 0 & -8 \end{bmatrix}.$$

These three row operations are recorded in the matrix

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 3 & 2/3 & 1 \end{bmatrix}.$$

Therefore, the LU decomposition of A is

$$\begin{bmatrix} 3 & -1 & 2 \\ -3 & -2 & 10 \\ 9 & -5 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 3 & 2/3 & 1 \end{bmatrix} \begin{bmatrix} 3 & -1 & 2 \\ 0 & -3 & 12 \\ 0 & 0 & -8 \end{bmatrix}.$$

12. Let A be the matrix

$$A = \begin{bmatrix} -2 & 2 \\ 1 & -1 \end{bmatrix}$$

and suppose $\vec{u}(t)$ solves the differential equation

$$\frac{d\vec{u}}{dt} = A\vec{u}(t)$$

subject to the initial condition $\vec{u}(0) = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$. What happens to $\vec{u}(t)$ as $t \to \infty$?

Answer: As $t \to \infty$, $\vec{u}(t) \to \begin{bmatrix} 3 \\ 3 \end{bmatrix}$. To see this, we need to solve the differential equation; the first step is to diagonalize A. We find the eigenvalues of A as the roots of

$$\det(A - \lambda I) = \begin{vmatrix} -2-\lambda & 2 \\ 1 & -1-\lambda \end{vmatrix} = (-2-\lambda)(-1-\lambda) - 2 = \lambda^2 + 3\lambda = \lambda(\lambda + 3).$$

Thus, the eigenvalues of A are $\lambda_1 = 0$ and $\lambda_2 = -3$. Since the solution $\vec{u}(t)$ will involve the expressions $e^{\lambda_i t}$, we can see immediately that the term containing e^{-3t} won't matter as $t \to \infty$ and that the term containing $e^{0t} = 1$ will stay constant regardless of t. In particular, this means that $\vec{u}(t)$ will approach a multiple of the eigenvector corresponding to $\lambda_1 = 0$.

This eigenvector generates the nullspace of

$$A - 0I = A = \begin{bmatrix} -2 & 2 \\ 1 & -1 \end{bmatrix},$$

so it is $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

For the sake of completeness, the eigenvector corresponding to $\lambda_2 = -3$ generates the nullspace of

$$A + 3I = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix},$$

so it is $\vec{v}_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$.

Thus, A diagonalizes as

$$A = S\Lambda S^{-1}$$

with

$$S = \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad \Lambda = \begin{bmatrix} 0 & 0 \\ 0 & -3 \end{bmatrix}.$$

Hence,

$$\begin{aligned} \vec{u}(t) &= e^{At}\vec{u}(0) \\ &= Se^{\Lambda t}S^{-1}\vec{u}(0) \\ &= \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & e^{-3t} \end{bmatrix} \begin{bmatrix} 1/3 & 2/3 \\ -1/3 & 1/3 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \end{bmatrix} \\ &= \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & e^{-3t} \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ e^{-3t} \end{bmatrix} \\ &= \begin{bmatrix} 3 - 2e^{-3t} \\ 3 + e^{-3t} \end{bmatrix} \end{aligned}$$

Therefore, as $t \to \infty$, we see that $\vec{u}(t) \to \begin{bmatrix} 3 \\ 3 \end{bmatrix}$.