

Math 215 Final Exam Practice Problems

1. For each of the following statements, say whether it is true or false. If the statement is true, prove it. If false, give a counterexample.

- (a) If eigenvectors \vec{x} and \vec{y} correspond to distinct eigenvalues, then $\vec{x}^H \vec{y} = 0$.
- (b) Let A be an $m \times n$ matrix and let \vec{b} be a vector in \mathbb{R}^m . If $m < n$, then $A\vec{x} = \vec{b}$ has infinitely many solutions.
- (c) If A is an $m \times n$ real matrix, then the nullspace of A^T is the orthogonal complement of the column space of A .
- (d) If S and T are subspaces of \mathbb{R}^2 , then their union (i.e., the set of vectors which are in either S or T) is also a subspace of \mathbb{R}^2 .
- (e) Let S be a plane through the origin in \mathbb{R}^3 and let P be the matrix which projects onto the plane S . Then for any $\vec{v} \in \mathbb{R}^3$,

$$\|\vec{v}\|^2 = \|P\vec{v}\|^2 + \|\vec{v} - P\vec{v}\|^2.$$

2. Is the set of all orthogonal $n \times n$ real matrices a vector space?

3. Suppose the first row of A is 2, 3 and its eigenvalues are $i, -i$. Find A .

4. If \vec{x}_1, \vec{x}_2 are the columns of S , what are the eigenvalues and eigenvectors of

$$A = S \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} S^{-1} \quad \text{and} \quad B = S \begin{bmatrix} 3 & 4 \\ 0 & 1 \end{bmatrix} S^{-1}?$$

5. If A is an $n \times (n - 1)$ matrix with rank $n - 2$ and $\vec{b} \in \mathbb{R}^n$ such that $A\vec{v} = \vec{b}$ for some $\vec{v} \in \mathbb{R}^{n-1}$, what is the dimension of the space of *all* solutions to the equation $A\vec{x} = \vec{b}$?

- 6. (a) Prove that the eigenvalues of A are the eigenvalues of A^T .
- (b) If A and B are $n \times n$ real symmetric matrices, then AB and BA have the same eigenvalues. [HINT: Use part (a)]

7. Is there a real 2×2 matrix $A \neq I$ such that $A^3 = I$?

8. Suppose A is diagonalizable and that

$$p(\lambda) = \det(A - \lambda I) = c_n \lambda^n + c_{n-1} \lambda^{n-1} + \dots + c_1 \lambda + c_0.$$

Show that

$$p(A) = c_n A^n + c_{n-1} A^{n-1} + \dots + c_1 A + c_0 I$$

is the zero matrix.

NOTE: The fact that this is true for *any* matrix (regardless of whether it's diagonalizable) is called the *Cayley-Hamilton Theorem*.

9. Suppose $A = \vec{u}\vec{v}^T$ where $\vec{u}, \vec{v} \in \mathbb{R}^n$ and $\vec{v} \neq \vec{0}$.

- (a) What is the rank of A ?
- (b) Show that \vec{u} is an eigenvector of A . What is the corresponding eigenvalue?
- (c) What are the other eigenvalues of A ?
- (d) Compute the trace of A in two different ways.

10. Let V be an n -dimensional vector space and suppose $T : V \rightarrow V$ is a linear transformation such that the range of T is equal to the set of vectors that T sends to $\vec{0}$. In other words,

$$\{T(\vec{v}) : \vec{v} \in V\} = \{\vec{v} \in V : T(\vec{v}) = \vec{0}\}.$$

- (a) Show that n must be even.
(b) Give an example of such a T .
11. Find the LU decomposition of the matrix

$$\begin{bmatrix} 3 & -1 & 2 \\ -3 & -2 & 10 \\ 9 & -5 & 6 \end{bmatrix}.$$

12. Let A be the matrix

$$A = \begin{bmatrix} -2 & 2 \\ 1 & -1 \end{bmatrix}$$

and suppose $\vec{u}(t)$ solves the differential equation

$$\frac{d\vec{u}}{dt} = A\vec{u}(t)$$

subject to the initial condition $\vec{u}(0) = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$. What happens to $\vec{u}(t)$ as $t \rightarrow \infty$?