Math 215 Exam #2 Solutions

1. (12 points) For each of the following statements, say whether it is true or false. If the statement is true, prove it. If false, give a counterexample.

   (a) If $\mathbf{S}$ is a k-dimensional subspace of $\mathbb{R}^n$, then $\mathbf{S}^\perp$ has dimension $n-k$.

   Answer: True. Let A be an $m \times n$ matrix whose row space is equal to $\mathbf{S}$ (for example, if $\{\vec{v}_1, \ldots, \vec{v}_k\}$ is a basis for $\mathbf{S}$, then you could let A be the matrix whose rows are the $\vec{v}_i^T$).

   Then certainly
$$k = \dim \mathbf{S} = \dim \operatorname{row}(A) = \operatorname{rank}(A).$$
   Moreover, since $\operatorname{nul}(A)$ is the orthogonal complement of $\operatorname{row}(A)$, we see that
$$\mathbf{S}^\perp = (\operatorname{row}(A))^\perp = \operatorname{nul}(A).$$
   Then we see that
$$\dim \mathbf{S}^\perp = \dim \operatorname{nul}(A) = n - \operatorname{rank}(A) = n - k.$$

   (b) If A is a 2×2 matrix all of whose entries are 0, 1, or -1, then $|\det A| \leq \sqrt{2}$.

   Answer: False. Consider the matrix
$$A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}.$$
   Then all the entries of A are 0, 1, or -1, but $\det A = 2$, which is definitely bigger than $\sqrt{2}$.

   (c) If P is a projection matrix (meaning that $P^2 = P$) with at least one non-zero eigenvalue, then 1 must be an eigenvalue of P.

Answer: True. Let $\lambda \neq 0$ be a non-zero eigenvalue of P, and let $\vec{v}$ be an eigenvector corresponding to λ. By definition, $P\vec{v} = \lambda\vec{v}$. Now, on the one hand,
$$P^2\vec{v} = P(P\vec{v}) = P(\lambda\vec{v}) = \lambda^2\vec{v}.$$
On the other hand,
$$P^2\vec{v} = P\vec{v} = \lambda\vec{v}$$
since $P^2 = P$. Therefore, we see that
$$\lambda^2 = \lambda.$$
Since $\lambda \neq 0$, this implies that $\lambda = 1$.

2. (8 points) Let $y_1, \ldots, y_n$ be a collection of measurements. Show that the horizontal line (i.e., $y = C$ for some C) which best fits the data (in the sense of least-squares) is the average of the measurements:

$$C = \frac{y_1 + \ldots + y_n}{n}.$$

Proof. If each data point y_i actually lay on the same horizontal line $y = C$, then we would have the system of equations

$$\begin{aligned} C &= y_1 \\ C &= y_2 \\ \vdots\ &\ \ \vdots \\ C &= y_n, \end{aligned}$$

or, equivalently,

$$\begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} [C] = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}.$$

Of course, this is almost certainly not true, so the above matrix equation is almost certainly not solvable. However, if we let $A = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$ and $\vec{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$, then the choice of C that makes $A[C]$ as close as possible to $\vec{y}$ will be

$$[C] = (A^T A)^{-1} A^T \vec{y}.$$

Now,

$$A^T A = [1\ 1\ \cdots\ 1] \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = [n],$$

so $(A^T A)^{-1} = [1/n]$. Therefore,

$$\begin{aligned} [C] &= (A^T A)^{-1} A^T \vec{y} \\ &= [1/n][1\ 1\ \cdots\ 1] \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \\ &= [1/n][y_1 + y_2 + \ldots + y_n] \\ &= \left[\frac{y_1 + y_2 + \ldots + y_n}{n} \right], \end{aligned}$$

just as desired.

$\square$

3. (16 points) Let A be the 2×2 matrix

$$A = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}.$$

(a) Show that -1 and 2 are eigenvalues of A.

Answer: Let's solve the characteristic equation:

$$\begin{aligned} 0 = \det(A - \lambda I) &= \begin{vmatrix} -\lambda & 1 \\ 2 & 1-\lambda \end{vmatrix} \\ &= -\lambda(1-\lambda) - 2 \\ &= \lambda^2 - \lambda - 2 \\ &= (\lambda - 2)(\lambda + 1), \end{aligned}$$

so the solutions are $\lambda_1 = 2$ and $\lambda_2 = -1$. To double-check (and because we'll need them in a minute anyway) we find the eigenvectors. The eigenvector $\vec{v}_1$ corresponding to $\lambda_1 = 2$ will be the generator of the nullspace of

$$A - 2I = \begin{bmatrix} -2 & 1 \\ 2 & -1 \end{bmatrix}.$$

The nullspace consists of multiples of $\begin{bmatrix} 1/2 \\ 1 \end{bmatrix}$; a particularly nice choice is $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

The eigenvector $\vec{v}_2$ corresponding to $\lambda_2 = -1$ will be the generator of the nullspace of

$$A - (-1)I = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}.$$

The nullspace consists of multiples of $\vec{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

Now, finally, we see that

$$A\vec{v}_1 = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix} = 2\vec{v}_1,$$

so $\lambda_1 = 2$ is indeed an eigenvalue of A.

Likewise

$$A\vec{v}_1 = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} = -\vec{v}_1,$$

so $\lambda_2 = -1$ is indeed an eigenvalue of A.

(b) Use (a) to diagonalize A. That is, find an invertible matrix S and a diagonal matrix Λ such that $A = S\Lambda S^{-1}$ (equivalently, $\Lambda = S^{-1}AS$). What is Λ^k?

Answer: We know that Λ is the eigenvalue matrix

$$\Lambda = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$$

and S is the eigenvector matrix

$$S = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}.$$

Notice that $\det S = 3$ and so

$$S^{-1} = \frac{1}{3}\begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 1/3 & 1/3 \\ -2/3 & 1/3 \end{bmatrix}.$$

Finally,

$$\Lambda^k = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}^k = \begin{bmatrix} 2^k & 0 \\ 0 & (-1)^k \end{bmatrix}.$$

(c) Let $x_0 = 0$ and $x_1 = 1$ and form the sequence $x_0, x_1, x_2, \ldots$ by letting

$$x_{k+1} = x_k + 2x_{k-1}.$$

Show that

$$\begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}^k \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} x_k \\ x_{k+1} \end{bmatrix}$$

for any $k \geq 1$.

Proof. Notice that

$$\begin{bmatrix} x_k \\ x_{k+1} \end{bmatrix} = \begin{bmatrix} x_k \\ x_k + 2x_{k-1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_{k-1} \\ x_k \end{bmatrix}.$$

We can, of course, continue the process, as the above is equal to:

$$\begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_{k-1} \\ x_{k-1} + 2x_{k-2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} \left(\begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_{k-2} \\ x_{k-1} \end{bmatrix} \right) = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}^2 \begin{bmatrix} x_{k-2} \\ x_{k-1} \end{bmatrix}.$$

By iterating this process, we see that, for any i,

$$\begin{bmatrix} x_k \\ x_{k+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}^i \begin{bmatrix} x_{k-i} \\ x_{k-i+1} \end{bmatrix}$$

and, in particular, when we let $i = k$

$$\begin{bmatrix} x_k \\ x_{k+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}^k \begin{bmatrix} x_0 \\ x_1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}^k \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

since we know that $x_0 = 0$ and $x_1 = 1$.

(NOTE: The above argument is completely equivalent to a more standard proof by induction, which would also, of course, be a valid way of approaching this problem.) □

(d) Use (b) and (c) to find a formula for x_n. Feel free to use the conclusion of (c) even if you had trouble proving it.

Answer: From (c) we know that, setting $k = n - 1$,

$$\begin{bmatrix} x_{n-1} \\ x_n \end{bmatrix} = A^{n-1} \begin{bmatrix} 0 & 1 \end{bmatrix}$$

Therefore, by part (b),

$$\begin{bmatrix} x_{n-1} \\ x_n \end{bmatrix} = A^{n-1} \begin{bmatrix} 0 & 1 \end{bmatrix} = \left(S\Lambda S^{-1}\right)^{n-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = S\Lambda^{n-1}S^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Hence, we compute that

$$\begin{aligned} \begin{bmatrix} x_{n-1} \\ x_n \end{bmatrix} &= \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2^{n-1} & 0 \\ 0 & (-1)^{n-1} \end{bmatrix} \begin{bmatrix} 1/3 & 1/3 \\ -2/3 & 1/3 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2^{n-1} & 0 \\ 0 & (-1)^{n-1} \end{bmatrix} \begin{bmatrix} 1/3 \\ 1/3 \end{bmatrix} \\ &= \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} \frac{2^{n-1}}{3} \\ \frac{(-1)^{n-1}}{3} \end{bmatrix} \\ &= \begin{bmatrix} \frac{2^{n-1}-(-1)^{n-1}}{3} \\ \frac{2\cdot 2^{n-1}+(-1)^{n-1}}{3} \end{bmatrix}. \end{aligned}$$

Therefore, since $2 \cdot 2^{n-1} = 2^n$ and since $(-1)^{n-1} = -(-1)^n$, we can see that

$$x_n = \frac{2^n - (-1)^n}{3}.$$