

## Math 215 Exam #2 Practice Problems

1. For each of the following statements, say whether it is true or false. If the statement is true, prove it. If false, give a counterexample.

- (a) If  $Q$  is an orthogonal matrix, then  $\det Q = 1$ .
- (b) Every invertible matrix can be diagonalized.
- (c) Every diagonalizable matrix is invertible.
- (d) If the matrix  $A$  is not invertible, then  $0$  is an eigenvalue of  $A$ .
- (e) If  $\vec{v}$  and  $\vec{w}$  are orthogonal and  $P$  is a projection matrix, then  $P\vec{v}$  and  $P\vec{w}$  are also orthogonal.
- (f) Suppose  $A$  is an  $n \times n$  matrix and that there exists some  $k$  such that  $A^k = 0$  (such matrices are called *nilpotent* matrices). Then  $A$  is not invertible.

2. Let  $Q$  be an  $n \times n$  orthogonal matrix. Show that if  $\{\vec{v}_1, \dots, \vec{v}_n\}$  is an orthonormal basis for  $\mathbb{R}^n$ , then so is  $\{Q\vec{v}_1, \dots, Q\vec{v}_n\}$ .

3. Let

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}.$$

- (a) Let  $R$  be the region in the plane enclosed by the unit circle. If  $T$  is the linear transformation of the plane whose matrix is  $A$ , what is the area of  $T(R)$ ?
- (b) Find the matrix for the transformation  $T^{-1}$  *without* doing elimination.

4. Let  $\ell$  be the line in  $\mathbb{R}^3$  through the vector  $\vec{a} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ .

(a) Find a basis for the orthogonal complement of  $\ell$ .

(b) If  $\vec{v} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$ , write  $\vec{v}$  as a sum

$$\vec{v} = \vec{v}_1 + \vec{v}_2,$$

where  $\vec{v}_1 \in \ell$  and  $\vec{v}_2 \in \ell^\perp$ .

5. Find the line  $C + Dt$  that best fits the data

$$(-1, 1), (0, 1), (1, 2).$$

6. Let  $\ell$  be the line through a vector  $\vec{a} \in \mathbb{R}^n$  and let  $P$  be the matrix which projects everything in  $\mathbb{R}^n$  to  $\ell$ .

- (a) Show that the trace of  $P$  equals 1.
- (b) What can you say about the eigenvalues of  $P$ ?

7. Suppose  $A$  is a  $2 \times 2$  matrix with eigenvalues  $\lambda_1$  and  $\lambda_2$  corresponding to non-zero eigenvectors  $\vec{v}_1$  and  $\vec{v}_2$ , respectively. If  $\lambda_1 \neq \lambda_2$ , show that  $\vec{v}_1$  and  $\vec{v}_2$  are linearly independent.