Math 215 Exam #1 Practice Problem Solutions

1. For each of the following statements, say whether it is true or false. If the statement is true, prove it. If false, give a counterexample.

   (a) If A is a 2×2 matrix such that $A(Ax) = 0$ for all $x \in \mathbb{R}^2$, then A is the zero matrix.
   Answer: False. If $A(Ax) = 0$ for all x, then the column space of A and the nullspace of A must be the same space. In particular, consider the matrix

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

   Then, for any $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2$, we have that

$$Ax = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ 0 \end{bmatrix}$$

   and

$$A(Ax) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

   Hence, $A(Ax) = 0$ for all x, but $A \neq 0$, so A gives a counterexample to the statement.

   (b) A system of 3 equations in 4 unknowns can never have a unique solution.
   Answer: True. We can realize such an system of equations as a single matrix equation

$$A\mathbf{x} = \mathbf{b},$$

   where A is a 3×4 matrix. Hence, $\operatorname{rank}(A) \leq 3$, so the dimension of the nullspace of A is at least 1:

$$\dim \operatorname{nul}(A) = 4 - \operatorname{rank}(A) \geq 4 - 3 = 1.$$

   Hence, there must be at least one free variable in the system, meaning that, if the system is solvable at all, it must have an infinite number of solutions.

   (c) If V is a vector space and S is a finite set of vectors in V, then some subset of S forms a basis for V.
   Answer: False. Let $V = \mathbb{R}^2$, which is clearly a vector space, and let S be the singleton set $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$. The single element of S does not span $\mathbb{R}^2$, so no subset of S can be a basis for $\mathbb{R}^2$. Hence, this provides a counterexample to the statement.

   (d) Suppose A is an $m \times n$ matrix such that $A\mathbf{x} = \mathbf{b}$ can be solved for any choice of $\mathbf{b} \in \mathbb{R}^m$. Then the columns of A form a basis for $\mathbb{R}^m$.
   Answer: False. Consider the matrix

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

   Then A is already in reduced echelon form and clearly has 2 pivots, so $\operatorname{rank}(A) = 2$. This implies that $\dim \operatorname{col}(A) = 2$, so the column space of A consists of all of $\mathbb{R}^2$. Thus, the equation $A\mathbf{x} = \mathbf{b}$ can be solved for any $\mathbf{b} \in \mathbb{R}^2$ (since any $\mathbf{b}$ is in $\operatorname{col}(A)$). However, the columns of A are clearly not linearly independent (no set containing the zero vector can be linearly independent), so they cannot form a basis for $\mathbb{R}^2$.

   A related but true statement would be the following: "Suppose A is an $m \times n$ matrix such that $A\mathbf{x} = \mathbf{b}$ can be solved for any choice of $\mathbf{b} \in \mathbb{R}^m$. Then *some subset of* the columns of A forms a basis for $\mathbb{R}^m$."

(e) Given 3 equations in 4 unknowns, each describes a hyperplane in $\mathbb{R}^4$. If the system of those 3 equations is consistent, then the intersection of the hyperplanes contains a line.

Answer: True. This is really just a restatement of (b). Translating the system of equations into a matrix equation $A\mathbf{x} = \mathbf{b}$, the nullspace of A must be at least one-dimensional, so the solution-space must be at least one-dimensional. Since the solution space of the matrix equation corresponds to the intersection of the hyperplanes, that intersection must be at least one-dimensional, meaning it must contain a line.

(f) If A is a symmetric matrix (i.e. $A = A^T$), then A is invertible.

Answer: False. Consider the symmetric matrix

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

Then A only has rank 1, meaning that A cannot be invertible, so this gives a counterexample to the statement.

(g) If $m < n$ and A is an $m \times n$ matrix such that $A\mathbf{x} = \mathbf{b}$ has a solution for all $\mathbf{b} \in \mathbb{R}^m$, then there exists $\mathbf{z} \in \mathbb{R}^m$ such that $A\mathbf{x} = \mathbf{z}$ has infinitely many solutions.

Answer: True. The fact that $A\mathbf{x} = \mathbf{b}$ has a solution for all $\mathbf{b} \in \mathbb{R}^m$ means that the column space of A is equal to all of $\mathbb{R}^m$. Hence,

$$\text{rank}(A) = \dim \text{col}(A) = m.$$

Since

$$\dim \text{nul}(A) = n - \text{rank}(A) = n - m$$

and since $m < n$, we have that the nullspace of A has some positive dimension. Since the nullspace of A consists precisely of those $\mathbf{x} \in \mathbb{R}^n$ such that $A\mathbf{x} = \mathbf{0}$, this equation has infinitely many solutions. Thus, letting $\mathbf{z} = \mathbf{0}$, we see that the statement is true.

(h) The set of polynomials of degree ≤ 5 forms a vector space.

Answer: True. You should check that the set of polynomials of degree ≤ 5 satisfies all the rules for being a vector space. The important facts are this space is closed under addition and scalar multiplication.

2. For each of the following, determine whether the given subset is a subspace of the given vector space. Explain your answer.

(a) **Vector Space:** $\mathbb{R}^4$.

Subset: The vectors of the form

$$\begin{bmatrix} a \\ b \\ 0 \\ d \end{bmatrix}.$$

Answer: Yes, this is a subspace. If we take two vectors in the subset, say $\begin{bmatrix} a_1 \\ b_1 \\ 0 \\ d_1 \end{bmatrix}$ and $\begin{bmatrix} a_2 \\ b_2 \\ 0 \\ d_2 \end{bmatrix}$, then their sum

$$\begin{bmatrix} a_1 \\ b_1 \\ 0 \\ d_1 \end{bmatrix} + \begin{bmatrix} a_2 \\ b_2 \\ 0 \\ d_2 \end{bmatrix} = \begin{bmatrix} a_1 + a_2 \\ b_1 + b_2 \\ 0 \\ d_1 + d_2 \end{bmatrix}$$

is also in the subset, so this set is closed under addition.

Moreover, if $c \in \mathbb{R}$, then

$$c \begin{bmatrix} a_1 \\ b_1 \\ 0 \\ d_1 \end{bmatrix} = \begin{bmatrix} ca_1 \\ cb_1 \\ 0 \\ cd_1 \end{bmatrix}$$

is in the set, so this set is closed under scalar multiplication.

Thus, the set is closed under both addition and scalar multiplication, and so is a subspace.

(b) **Vector Space:** $\mathbb{R}^2$.

Subset: The solutions to the equation $2x - 5y = 11$.

Answer: No, this is not a subspace. To see why, I'll show that it is not closed under addition. The vectors

$$\begin{bmatrix} \frac{11}{2} \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 \\ -\frac{11}{5} \end{bmatrix}$$

are both in the set, since the pairs $(11/2, 0)$ and $(0, -11/5)$ both solve the equation $2x - 5y = 11$, but

$$\begin{bmatrix} \frac{11}{2} \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -\frac{11}{5} \end{bmatrix} = \begin{bmatrix} \frac{11}{2} \\ -\frac{11}{5} \end{bmatrix}$$

is not in the set, since

$$2(11/2) - 5(-11/5) = 11 + 11 = 22.$$

Therefore, the set is not closed under addition, and so is not a subspace.

(c) **Vector Space:** $\mathbb{R}^n$.

Subset: All $\mathbf{x} \in \mathbb{R}^n$ such that $A\mathbf{x} = 2x$ where A is a given $n \times n$ matrix.

Answer: Yes, this is a subspace. To prove it, suppose $\mathbf{x}_1$ and $\mathbf{x}_2$ are in this set, meaning that

$$A\mathbf{x}_1 = 2\mathbf{x}_1 \quad \text{and} \quad A\mathbf{x}_2 = 2\mathbf{x}_2$$

(such vectors are called *eigenvectors* of A; we'll learn more about them later). Then

$$A(\mathbf{x}_1 + \mathbf{x}_2) = A\mathbf{x}_1 + A\mathbf{x}_2 = 2\mathbf{x}_1 + 2\mathbf{x}_2 = 2(\mathbf{x}_1 + \mathbf{x}_2),$$

meaning that $\mathbf{x}_1 + \mathbf{x}_2$ is in this set as well.

Moreover, for any $c \in \mathbb{R}$,

$$A(c\mathbf{x}_1) = c(A\mathbf{x}_1) = c(2\mathbf{x}_1) = 2(c\mathbf{x}_1),$$

so $c\mathbf{x}_1$ is in the set as well.

Therefore, this set is closed under addition and scalar multiplication, so it is indeed a subspace.

(d) **Vector Space:** $\mathbb{R}^3$.

Subset: The intersection of P_1 and P_2, where P_1 and P_2 are planes through the origin.

Answer: Yes, this is a subspace. The proof is essentially the same as you gave for Problem 3(c) from HW 4.

(e) **Vector Space:** All polynomials.

Subset: The quadratic (i.e. degree 2) polynomials.

Answer: No, this is not a subspace. To see that it is not closed under addition, notice that if $f(t) = t^2$ and $g(t) = -t^2$, then f and g are both in the set of quadratic polynomials, but, since

$$(f + g)(t) = f(t) + g(t) = t^2 + (-t^2) = 0,$$

the sum $f + g$ is not a quadratic polynomial.

(f) **Vector Space:** All real-valued functions.
Subset: Functions of the form $f(t) = a\cos t + b\sin t + c$ for $a, b, c \in \mathbb{R}$.
Answer: Yes, this is a subspace. If $a_1, a_2, b_1, b_2, c_1, c_2 \in \mathbb{R}$ and I define

$$f(t) = a_1 \cos t + b_1 \sin t + c_1$$

and

$$g(t) = a_2 \cos t + b_2 \sin t + c_2,$$

then f and g are in the given subset. The sum has the form

$$f(t)+g(t) = (a_1 \cos t+b_1 \sin t+c_1)+(a_2 \cos t+b_2 \sin t+c_2) = (a_1+a_2)\cos t+(b_1+b_2)\sin t+(c_1+c_2),$$

so $f + g$ is also in the subset, which is, therefore, closed under addition.
Also, if $r \in \mathbb{R}$, then

$$rf(t) = r(a_1 \cos t + b_1 \sin t + c_1) = (ra_1)\cos t + (rb_1)\sin t + (rc_1),$$

so rf is in the subset, which is, therefore, closed under scalar multiplication.
Hence, we can conclude that this subset is actually a subspace.

3. Consider the matrix

$$A = \begin{bmatrix} 1 & a \\ a & 1 \end{bmatrix}.$$

(a) Under what conditions on a is A invertible?
Answer: The matrix A is invertible if and only if it has rank 2. To see what the rank is, we do elimination. The first step is to subtract a times row 1 from row 2, yielding

$$\begin{bmatrix} 1 & a \\ 0 & 1-a^2 \end{bmatrix}.$$

Then this has a second pivot if and only if $1 - a^2 \neq 0$, meaning that $a^2 \neq 1$, or $a \neq \pm 1$. Thus, A is invertible so long as a is neither 1 nor -1.

(b) Choose a non-zero value of a that makes A invertible and determine A^{-1}.
Answer: Choose $a = 2$. Recall that we can find the inverse of A by converting the left side of the following augmented matrix to the identity:

$$\begin{bmatrix} 1 & 2 & & 1 & 0 \\ 2 & 1 & & 0 & 1 \end{bmatrix}.$$

Subtract twice row 1 from row 2:

$$\begin{bmatrix} 1 & 2 & & 1 & 0 \\ 0 & -3 & & -2 & 1 \end{bmatrix}.$$

Scale the second row by $-\frac{1}{3}$ and also subtract twice the result from row 1:

$$\begin{bmatrix} 1 & 0 & & -\frac{1}{3} & \frac{2}{3} \\ 0 & 1 & & \frac{2}{3} & -\frac{1}{3} \end{bmatrix}.$$

Therefore,

$$A^{-1} = \begin{bmatrix} -\frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} \end{bmatrix}.$$

(c) For each value of a that makes A non-invertible, determine the dimension of the nullspace of A.
Answer: When $a = 1$, the matrix

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix},$$

which, after subtracting row 1 from row 2, reduces to

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}.$$

Hence, A has rank 1, so the nullspace has dimension

$$\dim \operatorname{nul}(A) = 2 - \operatorname{rank}(A) = 2 - 1 = 1.$$

4. Consider the system of equations

$$\begin{array}{rcrcrcrcl} x_1 & + & 2x_2 & + & x_3 & - & 3x_4 & = & b_1 \\ x_1 & + & 2x_2 & + & 2x_3 & - & 5x_4 & = & b_2 \\ 2x_1 & + & 4x_2 & + & 3x_3 & - & 8x_4 & = & b_3 \end{array}.$$

(a) Find all solutions when the above system is homogeneous (i.e. $b_1 = b_2 = b_3 = 0$). Find a basis for the space of solutions to the homogeneous system.
Answer: Convert the system into the augmented matrix

$$\begin{bmatrix} 1 & 2 & 1 & -3 & 0 \\ 1 & 2 & 2 & -5 & 0 \\ 2 & 4 & 3 & -8 & 0 \end{bmatrix}.$$

Now do elimination to get the reduced echelon form. First, subtract row 1 from row 2 and subtract twice row 1 from row 3:

$$\begin{bmatrix} 1 & 2 & 1 & -3 & 0 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & -2 & 0 \end{bmatrix}.$$

Now, subtract row 2 from both row 1 and row 3:

$$\begin{bmatrix} 1 & 2 & 0 & -1 & 0 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then this system is consistent provided that

$$\begin{aligned} x_1 &= -2x_2 + x_4 \\ x_3 &= 2x_4. \end{aligned}$$

. Hence, the solutions to the homogeneous equation are those vectors of the form

$$x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \end{bmatrix}$$

for $x_2, x_4 \in \mathbb{R}$. Then a basis for the space of solutions to the homogeneous system (i.e. nullspace of the corresponding matrix) is

$$\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \end{bmatrix} \right\}.$$

(b) Let S be the set of vectors $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ such that the system can be solved. What is the dimension of S?

Answer: Letting A be the matrix of the system, we know that the set of vectors $\mathbf{b}$ for which the system can be solved is the column space of A. Since A is 3×4, we know that

$$\operatorname{rank}(A) + \dim \operatorname{nul}(A) = 4.$$

Since, from part (a), we know that the dimension of the nullspace is 2, this implies that the column space of A is two-dimensional.

(c) It's easy to check that the vector $\mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \end{bmatrix}$ is a solution to the system that arises when $b_1 = 3$, $b_2 = 5$, and $b_3 = 8$. Find *all* the solutions to this system.

Answer: All solutions $\mathbf{x}$ to the system $A\mathbf{x} = \mathbf{b}$ take the form $\mathbf{x} = \mathbf{x}_p + \mathbf{x}_h$, where $\mathbf{x}_p$ is a particular solution and $\mathbf{x}_h$ is the homogeneous solution to the corresponding homogeneous problem. Thus, we can let $\mathbf{x}_p = \mathbf{v}$, which we're told solves the system and we see that, using part (a), the general solution is

$$\begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \end{bmatrix},$$

where $x_2, x_4 \in \mathbb{R}$.