

Math 115 Exam #4 Practice Problem Solutions

1. Solve the initial-value problem

$$y'' + 8y' + 16y = 0, \quad y(0) = 3, \quad y'(0) = 6.$$

Answer: The characteristic equation is

$$r^2 + 8r + 16 = 0,$$

which factors as

$$(r + 4)^2 = 0,$$

so $r = -4$ is the only solution of the equation. Therefore, the general solution to the differential equation is

$$y = C_1e^{-4x} + C_2xe^{-4x}.$$

Plugging in $x = 0$, we get

$$3 = y(0) = C_1e^0 + C_2(0)e^0 = C_1,$$

so

$$y = 3e^{-4x} + C_2xe^{-4x}.$$

Hence,

$$y' = -12e^{-4x} + C_2e^{-4x} - 4C_2xe^{-4x}.$$

Therefore,

$$6 = y'(0) = -12 + C_2 + 0 = -12 + C_2,$$

or, equivalently, $C_2 = 18$. Thus, the solution is

$$y = 3e^{-4x} + 18xe^{-4x}.$$

2. Solve the differential equation

$$y'' + 2y' + 37y = 0.$$

Answer: The characteristic equation is

$$r^2 + 2r + 37 = 0.$$

Using the quadratic formula, solutions are of the form

$$\frac{-2 \pm \sqrt{4 - 4(1)(37)}}{2} = \frac{-2 \pm \sqrt{-144}}{2} = -1 \pm 6i.$$

Therefore, the solution to the differential equation is

$$y = C_1e^{-x} \cos 6x + C_2e^{-x} \sin 6x.$$

3. Solve the initial-value problem

$$y'' + 5y' - 24y = 0, \quad y(0) = 0, \quad y'(0) = 3.$$

Answer: The characteristic equation is

$$r^2 + 5r - 24 = 0,$$

which factors as

$$(r + 8)(r - 3) = 0,$$

so the solutions to the characteristic equation are $r_1 = -8$, $r_2 = 3$. Hence, the solution the homogeneous equation is

$$y = C_1 e^{-8x} + C_2 e^{3x}.$$

Plugging in $x = 0$, we get

$$0 = y(0) = C_1 + C_2,$$

so $C_1 = -C_2$. Hence

$$y = -C_2 e^{-8x} + C_2 e^{3x}$$

and so

$$y' = 8C_2 e^{-8x} + 3C_2 e^{3x}.$$

Plugging in $x = 0$, we get

$$3 = y'(0) = 8C_2 + 3C_2 = 11C_2,$$

so $C_2 = 3/11$, meaning that $C_1 = -C_2 = -3/11$. Therefore, the solution to the initial-value problem is

$$y = -\frac{3}{11} e^{-8x} + \frac{3}{11} e^{3x}.$$

4. Solve the differential equation

$$y'' + 9y = e^{2x}.$$

Answer: First, solve the homogeneous equation $y'' + 9y = 0$. The characteristic equation is

$$r^2 + 9 = 0$$

meaning that $r^2 = -9$ and so $r = \pm 3i$. Therefore, the complementary solution is

$$y_c = C_1 \cos 3x + C_2 \sin 3x.$$

Now, to find a particular solution, guess that

$$y_p = Ae^{2x}.$$

Then $y'_p = 2Ae^{2x}$ and $y''_p = 4Ae^{2x}$. Therefore, since y_p solves the equation,

$$e^{2x} = y''_p + 9y_p = 4Ae^{2x} + 9Ae^{2x} = 13Ae^{2x}.$$

Therefore $A = 1/13$ and so

$$y_p = \frac{1}{13} e^{2x}.$$

Thus, the solution of the given equation is

$$y = y_c + y_p = C_1 \cos 3x + C_2 \sin 3x + \frac{1}{13} e^{2x}.$$

5. Solve the differential equation

$$y'' + 5y' + 6y = x^2$$

Answer: First, solve the homogeneous equation $y'' + 5y' + 6y = 0$. This has characteristic equation

$$r^2 + 5r + 6 = 0,$$

which factors as

$$(r + 3)(r + 2),$$

so the solutions are $r_1 = -3$, $r_2 = -2$. Thus, the complementary solution is

$$y_c = C_1 e^{-3x} + C_2 e^{-2x}.$$

Now, guess that the particular solution is

$$y_p = Ax^2 + Bx + C.$$

Then

$$y'_p = 2Ax + B$$

and

$$y''_p = 2A.$$

Hence, since y_p solves the equation,

$$x^2 = y''_p + 5y'_p + 6y_p = 2A + 5(2Ax + B) + 6(Ax^2 + Bx + C) = 6Ax^2 + (10A + 6B)x + (2A + 5B + 6C).$$

Therefore, $6A = 1$ and so $A = 1/6$. In turn,

$$0 = 10A + 6B = \frac{10}{6} + 6B,$$

so $6B = -10/6$ and $B = -10/36 = -5/18$. Finally,

$$0 = 2A + 5B + 6C = \frac{2}{6} - \frac{25}{18} + 6C,$$

so we have that $6C = 19/18$, meaning that $C = 19/108$. Thus, the particular solution is

$$y_p = \frac{x^2}{6} - \frac{5}{18}x + \frac{19}{108}.$$

In turn, the general solution is

$$y = y_c + y_p = C_1 e^{-3x} + C_2 e^{-2x} + \frac{x^2}{6} - \frac{5}{18}x + \frac{19}{108}.$$

6. Solve the differential equation

$$y'' + 6y' + 9y = 1 + x$$

Answer: First, solve the homogeneous equation $y'' + 6y' + 9 = 0$. This has characteristic equation

$$r^2 + 6r + 9 = 0,$$

which factors as

$$(r + 3)^2 = 0.$$

Therefore, the solutions of the characteristic equation are $r = -3$, meaning that the complementary solution is

$$y_c = C_1 e^{-3x} + C_2 x e^{-3x}.$$

To find the particular solution, guess

$$y_p = Ax + B.$$

Then

$$y'_p = A$$

and

$$y''_p = 0.$$

Since y_p solves the equation, we have

$$1 + x = y''_p + 6y'_p + 9y_p = 0 + 6A + 9(Ax + B) = 9Ax + (6A + 9B).$$

Therefore,

$$1 = 9A,$$

so $A = 1/9$. Also,

$$1 = 6A + 9B = \frac{6}{9} + 9B,$$

so $9B = 1/3$, or $B = 1/27$. Therefore,

$$y_p = \frac{x}{9} + \frac{1}{27}.$$

Putting this together with the complementary solution, then,

$$y = y_c + y_p = C_1e^{-3x} + C_2xe^{-3x} + \frac{x}{9} + \frac{1}{27}.$$

7. Use power series to solve the differential equation

$$y'' + xy' + 2y = 0.$$

Answer: Suppose

$$y = \sum_{k=0}^{\infty} a_k x^k.$$

Then

$$y' = \sum_{k=1}^{\infty} k a_k x^{k-1}$$

and

$$y'' = \sum_{k=2}^{\infty} k(k-1)a_k x^{k-2}.$$

Since y is supposed to solve the given equation, we should have that

$$\begin{aligned} 0 &= y'' + xy' + 2y \\ 0 &= \sum_{k=2}^{\infty} k(k-1)a_k x^{k-2} + x \sum_{k=1}^{\infty} k a_k x^{k-1} + 2 \sum_{k=0}^{\infty} a_k x^k \\ 0 &= \sum_{k=2}^{\infty} k(k-1)a_k x^{k-2} + \sum_{k=1}^{\infty} k a_k x^k + \sum_{k=0}^{\infty} 2a_k x^k \\ 0 &= \sum_{k=0}^{\infty} (k+2)(k+1)a_{k+2} x^k + \sum_{k=1}^{\infty} k a_k x^k + \sum_{k=0}^{\infty} 2a_k x^k \\ 0 &= 2a_2 + 2a_0 + \sum_{k=1}^{\infty} [(k+2)(k+1)a_{k+2} + (k+2)a_k] x^k. \end{aligned}$$

Therefore,

$$0 = 2a_2 + 2a_0 \implies a_2 = -a_0$$

$$0 = (k+2)(k+1)a_{k+2} + (k+2)a_k \implies a_{k+2} = -\frac{1}{k+1}a_k$$

Hence

$$\begin{aligned} a_3 &= -\frac{1}{2}a_1 \\ a_4 &= -\frac{1}{3}a_2 = \frac{1}{3}a_0 \\ a_5 &= -\frac{1}{4}a_3 = \frac{1}{2 \cdot 4}a_1 \\ a_6 &= -\frac{1}{5}a_4 = -\frac{1}{3 \cdot 5}a_0 \\ &\vdots \end{aligned}$$

Putting this all together,

$$y = a_0 \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{1 \cdot 3 \cdot 5 \cdots (2k-1)} + a_1 \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2^k k!}$$

8. Use power series to solve the differential equation

$$y' = 4x^2 y.$$

[Hint: your solution should be the power series of some recognizable function.]

Answer: Re-write the equation as $y' - 4x^2 y = 0$. Assuming

$$y = \sum_{k=0}^{\infty} a_k x^k,$$

we have that

$$y' = \sum_{k=1}^{\infty} k a_k x^{k-1}.$$

Then, since y is supposed to solve the equation,

$$\begin{aligned} 0 &= y' - 4x^2 y \\ 0 &= \sum_{k=1}^{\infty} k a_k x^{k-1} - 4x^2 \sum_{k=0}^{\infty} a_k x^k \\ 0 &= \sum_{k=0}^{\infty} (k+1) a_{k+1} x^k - \sum_{k=0}^{\infty} 4a_k x^{k+2} \\ 0 &= \sum_{k=0}^{\infty} (k+1) a_{k+1} x^k - \sum_{k=2}^{\infty} 4a_{k-2} x^k \\ 0 &= a_1 + 2a_2 x + \sum_{k=2}^{\infty} [(k+1) a_{k+1} - 4a_{k-2}] x^k. \end{aligned}$$

Therefore, $a_1 = 0$ and $a_2 = 0$, while for $k \geq 2$,

$$0 = (k+1)a_{k+1} - 4a_{k-2} \implies a_{k+1} = \frac{4}{k+1}a_{k-2}.$$

Hence,

$$\begin{aligned} a_3 &= \frac{4}{3}a_0 \\ a_4 &= \frac{4}{4}a_1 = 0 \\ a_5 &= \frac{4}{5}a_2 = 0 \\ a_6 &= \frac{4}{6}a_3 = \frac{4^2}{3 \cdot 6}a_0. \end{aligned}$$

Hence,

$$y = a_0 \sum_{k=0}^{\infty} \frac{4^k x^{3k}}{3^k k!}.$$

Notice that y can be written as

$$y = a_0 \sum_{k=0}^{\infty} \frac{\left(\frac{4x^3}{3}\right)^k}{k!} = a_0 e^{\frac{4x^3}{3}}.$$