

### Math 115 Exam #3 Solutions

1. Solve the initial-value problem  $\frac{dy}{dx} = xy + x$ ,  $y(0) = 10$ .

**Answer:** Factoring out an  $x$  on the left hand side yields

$$\frac{dy}{dx} = x(y + 1),$$

which is a separable equation. Separate variables and integrate:

$$\int \frac{dy}{y + 1} = \int x dx.$$

Therefore

$$\ln |y + 1| = \frac{x^2}{2} + C.$$

Exponentiating both sides:

$$|y + 1| = e^{x^2/2 + C} = Ae^{x^2/2}.$$

If we let  $A$  be either positive or negative, we can eliminate the absolute value signs, so

$$y + 1 = Ae^{x^2/2},$$

meaning that

$$y = Ae^{x^2/2} - 1.$$

Plugging in  $x = 0$  yields

$$10 = Ae^{0^2/2} - 1 = A - 1,$$

so  $A = 11$ . Therefore, the solution is

$$y = 11e^{x^2/2} - 1.$$

2. Solve the initial-value problem  $(x^2 + 1)y' + 3xy = 6x$ ,  $y(0) = 0$ .

**Answer:** Dividing everything by  $x^2 + 1$  yields the linear equation in standard form

$$y' + \frac{3x}{x^2 + 1}y = \frac{6x}{x^2 + 1}.$$

Here  $P(x) = \frac{3x}{x^2 + 1}$  and  $Q(x) = \frac{6x}{x^2 + 1}$ , so

$$\int P(x)dx = \int \frac{3x}{x^2 + 1}dx = \frac{3}{2} \ln |x^2 + 1| = \ln |x^2 + 1|^{3/2} = \ln(x^2 + 1)^{3/2}$$

(integration by letting  $u = x^2 + 1$ , absolute value signs eliminated since  $x^2 + 1 > 0$  for all  $x$ ). Therefore,

$$\mu(x) = e^{\int P(x)dx} = e^{\ln(x^2 + 1)^{3/2}} = (x^2 + 1)^{3/2}.$$

Hence

$$\int \mu(x)Q(x)dx = \int (x^2 + 1)^{3/2} \frac{6x}{x^2 + 1} dx = \int \sqrt{x^2 + 1} \cdot 6x dx = 3 \cdot \frac{2}{3} (x^2 + 1)^{3/2} + C = 2(x^2 + 1)^{3/2} + C$$

and so

$$y = \frac{1}{\mu(x)} \int \mu(x)Q(x)dx = \frac{1}{(x^2 + 1)^{3/2}} \left( 2(x^2 + 1)^{3/2} + C \right) = 2 + \frac{C}{(x^2 + 1)^{3/2}}.$$

When  $x = 0$  we have that

$$0 = 2 + \frac{C}{(0 + 1)^{3/2}} = 2 + C,$$

so  $C = -2$ . Putting it all together, we see that

$$y = 2 - \frac{2}{(x^2 + 1)^{3/2}}.$$

3. Solve the initial-value problem  $x^4 \frac{dy}{dx} + 3x^3 y = xe^x$ ,  $x > 0$ ,  $y(1) = -1$ .

**Answer:** Dividing everything by  $x^4$  yields the following linear equation in standard form:

$$\frac{dy}{dx} + \frac{3}{x}y = \frac{e^x}{x^3}.$$

Here  $P(x) = \frac{3}{x}$  and  $Q(x) = \frac{e^x}{x^3}$ , so

$$\int P(x)dx = \int \frac{3}{x}dx = 3 \ln|x| = \ln|x|^3 = \ln(x^3)$$

since  $x > 0$ . Therefore,

$$\mu(x) = e^{\int P(x)dx} = e^{\ln(x^3)} = x^3.$$

In turn, this means that

$$\int \mu(x)Q(x)dx = \int x^3 \frac{e^x}{x^3}dx = \int e^x dx = e^x + C,$$

so

$$y = \frac{1}{\mu(x)} \int \mu(x)Q(x)dx = \frac{1}{x^3} (e^x + C) = \frac{e^x}{x^3} + \frac{C}{x^3}.$$

Plugging in  $x = 1$ , we see that

$$-1 = \frac{e^1}{1^3} + \frac{C}{1^3} = e + C,$$

so  $C = -1 - e$ , and so

$$y = \frac{e^x}{x^3} - \frac{1+e}{x^3}.$$

4. A super-fast-growing bacteria reproduces so quickly that the rate of production of new bacteria is proportional to the *square* of the number already present. If a sample starts with 100 bacteria, and after 3 hours there are 200 bacteria, how long (after the starting time) will it take until there are (theoretically) an *infinite* number of bacteria?

**Answer:** Let  $P(t)$  be the population of bacteria after  $t$  hours. Then the above says that

$$\frac{dP}{dt} = kP^2.$$

This is a separable equation, so we can separate variables and integrate:

$$\int \frac{dP}{P^2} = \int k dt.$$

Therefore,

$$-\frac{1}{P} = kt + C,$$

meaning that

$$P = -\frac{1}{kt + C}.$$

Plugging in  $t = 0$  gives

$$100 = -\frac{1}{k(0) + C} = -\frac{1}{C},$$

so  $C = -\frac{1}{100}$ . In turn, plugging in  $t = 3$  yields

$$200 = -\frac{1}{k(3) - \frac{1}{100}} = -\frac{1}{3k - \frac{1}{100}}.$$

Hence,

$$3k - \frac{1}{100} = -\frac{1}{200},$$

meaning that

$$3k = -\frac{1}{200} + \frac{1}{100} = \frac{1}{200}$$

and  $k = \frac{1}{600}$ . Therefore,

$$P(t) = -\frac{1}{\frac{t}{600} - \frac{1}{100}} = -\frac{600}{t-6}.$$

The population will theoretically be infinite when the denominator is zero, i.e. after 6 hours.

5. In the following predator-prey system, determine which of the variables,  $x$  or  $y$ , represents the prey population and which represents the predator population. Is the growth of the prey restricted just by the predators or by other factors as well? Explain.

$$\begin{aligned}\frac{dx}{dt} &= 0.2x - 0.0002x^2 - 0.006xy \\ \frac{dy}{dt} &= -0.015y + 0.00008xy.\end{aligned}$$

**Answer:** In this system, an increase in the population of  $y$  causes the rate of growth of  $x$  to decrease, whereas an increase in the population of  $x$  causes the rate of growth of  $y$  to increase. Thus, the behavior of the system is consistent with  $x$  being the prey and  $y$  being the predator.

The growth of the prey,  $x$ , is restricted by the presence of the predators,  $y$ , but is also restricted by an additional factor, indicated in the expression for  $\frac{dx}{dt}$  by the term  $-0.0002x^2$ . Notice that, as long as  $x$  isn't too big, the term  $0.2x$  dominates  $-0.0002x^2$ . However, as  $x$  gets very large, eventually  $-0.0002x^2$  will dominate  $0.2x$ . In other words, although for relatively small  $x$  increases in  $x$  cause an increase in  $\frac{dx}{dt}$ , the rate of growth of  $x$ , the rate of growth will decline as  $x$  gets bigger and bigger. Thus, a reasonable guess is that the additional factor restricting the growth of  $x$  is overcrowding.

In fact, we can re-write the first equation as

$$\frac{dx}{dt} = 0.2x \left( 1 - \frac{1}{1000}x \right) - 0.006xy,$$

which is the type of equation we derived when we assumed that, in the absence of predators, the prey population experienced logistic rather than exponential growth. Thus, the guess that the extra factor has to do with overcrowding (or, equivalently, carrying capacity), was correct.